

ANOMALOUS TRANSPORT IN MAGNETIZED SHEAR FLOW

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INTRODUCTION

After the initial stage of fast expanding of the hot Universe comes epoch of temperature fall and subsequent formation of dense clouds of hydrogen [Gamov (1955)]. Due to process of accretion some of this clouds became compact objects. Initially accretion is spherical, but then take the shape of the disk, and this 2D disk collect matter more efficient than 3D sphere. Accretion disks provide the mechanism of redistribution of angular momentum and extraction of potential energy, leads to occurrence of compact objects from gas clouds. To fall moving along a spiral trajectory on the central gravitating body, a particle rotating around must reduce angular momentum and energy. Outward of angular momentum and movement of gas particles in the opposite direction (to the central body) happens due to viscous friction. We can think that the disk is composed of multiple adjacent rings friction each other, each inner ring has a large angular velocity to the adjacent outer ring and thus friction of the drive ring decreased the rate of internal.

If the gravitating body with radius R has mass M then gravitational potential energy released in the process of accretion m will be [Frank et al. (2002)]

$$E_{acc} = \frac{GMm}{R}. \quad (1)$$

For example, a neutron star this energy would be about 10^{20} erg/g. Energy released for some cases of black holes can reach approximately 40% mc^2 , while the energy released in nuclear reaction $H \rightarrow He$ is of the order $E_{nuc} \approx 0.007mc^2$ or $6 \cdot 10^{18}$ erg/g.

Luminosity of the same accretion object is

$$L_{acc} = \frac{GM\dot{M}}{R}, \quad (2)$$

where \dot{M} is accretion rate. From observational data we have estimation for luminosity of quasars $L_{acc} \sim 10^{12} L_{\odot}$

The occurrence of intense dissipation in accretion flows is among the long-standing unsolved problems in astrophysics. [Balbus and Hawley Part I (1991), Fridman and Bisikalo (2008)] Revealing how the magnetized turbulence creates shear stress tensor is of primary importance to understand the heating mechanism and the transport of angular momentum in accretion disks. The transport of angular momentum at greatly enhanced rates is important for the main problem of cosmogony, that is understanding the dynamics of creation of compact astrophysical objects. [Balbus and Hawley (1998), Balbus (2003)] Without a theory explaining the enhanced energy dissipation in accretion flows of turbulent magnetized plasma we would have no clear picture

of how our solar system has been created, why the angular momentum of the Sun is only 2% of the angular momentum of solar system, while carrying 99% of the solar system's mass, why quasars are the most luminous sources in the universe. The importance of friction forces and convection as well the problem of angular momentum redistribution for the first time was emphasized by von Weizsäcker; [Weizsäcker (1948)] a very detailed bibliography on the physics of disks is provided in the monograph by Morozov and Khoperskov. [Morozov and Khoperskov (2005)] Gravitational forces, angular momentum conservation, and dissipation processes become the main ingredients of the standard model of accretion disks by Shakura and Sunyaev [Shakura and Sunayev (1973)] and Lynden-Bell and Pringle. [Lynden-Bell (1974)] Lynden-Bell [Lynden-Bell (1969)] suggested that quasars are accretion disks and Shakura and Sunyaev introduced alpha phenomenology for the stress tensor

$$\sigma_{R\varphi} = \alpha p, \quad p \sim \rho c_s^2, \quad (3)$$

where α is a dimensionless parameter, p is the pressure ρ is the mass density, c_s is the sound speed, and the indices of the tensor come from the cylindrical coordinate system (R, φ, z) related to the disk rotating around the z -axis. As accretion disks can have completely different scales for protostellar disks, mass transfer disks, and disks in active galactic nuclei (AGN) it is unlikely that Coriolis force is the main cause for dissipation, while the bending of the trajectories is a critical ingredient. We suppose that the shear dissipation in magnetized turbulent plasma is a robust and very general phenomenon which can be analyzed as a local heating for approximately homogeneous magnetic field and gradient of the velocity.

There is almost a consensus that the magnetic field is essential and should be introduced from the very beginning in the magnetohydrodynamic (MHD) analysis. The likely importance of MHD waves on the cosmogony of solar system has been pointed out by Alfvén: “At last some remarks are made about the transfer of momentum from the Sun to the planets, which is fundamental to the theory. The importance of the magnetohydrodynamic waves in this respect is pointed out. [Alfven (1946)]”

DERIVATION OF GENERAL SET OF EQUATIONS

1.1 Model and MHD Equations

Our starting point are the conservation laws for energy and momentum for an incompressible fluid with mass density ρ

$$\partial_t(\rho V_i) + \partial_k(\Pi_{ik}) = 0, \quad (1.1)$$

$$\frac{\partial}{\partial t} \left(\frac{\rho V^2}{2} + \rho \tilde{\varepsilon} + \frac{B^2}{2\mu_0} \right) + \operatorname{div} \mathbf{q} = 0, \quad (1.2)$$

$$\operatorname{div} \mathbf{V} = 0, \quad \rho = \text{const}, \quad (1.3)$$

where we have for total stress tensor Π and heat flux \mathbf{q} respectively

$$\Pi_{ik} = \rho V_i V_k + P \delta_{ik} - \eta \left(\frac{\partial V_i}{\partial x_k} + \frac{\partial V_k}{\partial x_i} \right) - \frac{1}{\mu_0} \left(B_i B_k - \frac{1}{2} \delta_{ik} B^2 \right), \quad (1.4)$$

$$\begin{aligned} \mathbf{q} = & \rho \left(\frac{V^2}{2} + \tilde{w} \right) \mathbf{V} - \mathbf{V} \cdot \boldsymbol{\sigma}' - \varkappa \nabla T + \frac{1}{\mu_0} [\mathbf{B} \times (\mathbf{V} \times \mathbf{B})] \\ & - \frac{\varepsilon_0 c^2 \varrho}{\mu_0} (\mathbf{B} \times \operatorname{curl} \mathbf{B}), \end{aligned} \quad (1.5)$$

where $\tilde{\varepsilon}$ is the internal energy per unit mass, \tilde{w} is the enthalpy per unit mass, $\sigma'_{ij} \equiv \eta(\partial_i V_j + \partial_j V_i)$ the viscous part of the stress tensor for an incompressible fluid, η is the viscosity, \varkappa is the heat conductivity, T is the temperature, ϱ is the Ohmic resistivity. The formula are written in SI, for a transition to Gaussian system we substitute $\mu_0 = 4\pi$ and $\varepsilon_0 = 1/4\pi$, i.e. expressions are written in invariant form.

For the magnetic field's energy density rate of change we have

$$\begin{aligned} \frac{1}{2\mu_0} \frac{\partial}{\partial t} B^2 &= \frac{1}{\mu_0} \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} = \frac{1}{\mu_0} \mathbf{B} \cdot [\nabla \times (\mathbf{V} \times \mathbf{B}) - \nu_m \nabla \times (\nabla \times \mathbf{B})] \\ &= \frac{1}{\mu_0} \nabla \cdot (\mathbf{B} \times (\mathbf{V} \times \mathbf{B}) - \nu_m \mathbf{B} \times (\nabla \times \mathbf{B})). \end{aligned} \quad (1.6)$$

We calculate the divergence of the total stress tensor Eq. (1.4), and using that

$$\begin{aligned} \partial_k \frac{1}{\mu_0} \left(B_i B_k - \frac{1}{2} B^2 \delta_{ik} \right) &= \frac{1}{\mu_0} \left(\mathbf{B} \cdot \nabla \mathbf{B} - \frac{1}{2} \nabla B^2 \right) \\ &= \frac{1}{\mu_0} (\mathbf{B} \times \text{curl } \mathbf{B}) = \mathbf{j} \times \mathbf{B}, \end{aligned} \quad (1.7)$$

we obtain the equation of motion for an incompressible plasma,

$$\rho (\partial_t \mathbf{V} + \mathbf{V} \cdot \nabla \mathbf{V}) = -\nabla P + \mathbf{j} \times \mathbf{B} + \eta \nabla^2 \mathbf{V}. \quad (1.8)$$

To close the system of equations we need to use Ampère's law, $\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$, and Ohm's law, $\mathbf{E} + \mathbf{V} \times \mathbf{B} = \varrho \mathbf{j}$, and supplement them with Faraday's law

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}. \quad (1.9)$$

For the second set of MHD equations we have

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left(\mathbf{V} \times \mathbf{B} - \frac{\varrho}{\mu_0} \nabla \times \mathbf{B} \right). \quad (1.10)$$

The MHD equations for an incompressible fluid $\rho = \text{const}$, in homogeneous magnetic field \mathbf{B}_0 , shear flow with rate A , angular velocity $\boldsymbol{\Omega} = \Omega \mathbf{e}_z$, and dimensionless angular velocity $\omega = \Omega/A$ are

$$\rho D_t \mathbf{V} = -\nabla P + \frac{1}{\mu_0} (\nabla \cdot \mathbf{B}) \mathbf{B} - \nabla \frac{B^2}{2\mu_0} - 2\rho \boldsymbol{\Omega} \times \mathbf{V} + \rho \nu_k \nabla^2 \mathbf{V}, \quad (1.11)$$

$$D_t \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{V} + \nu_m \nabla^2 \mathbf{B}, \quad \text{div } \mathbf{V} = 0, \quad \text{div } \mathbf{B} = 0, \quad (1.12)$$

where $D_t \equiv \partial_t + \mathbf{V} \cdot \nabla$ is the substantial (convective) derivative, P is the pressure, \mathbf{j} is the current density and ν_k is the kinematic viscosity, The magnetic diffusivity $\nu_m = \varepsilon_0 c^2 \varrho$ is expressed by the Ohmic resistance ϱ and $\varepsilon_0 = 1/\mu_0 c^2$. In order to obtain a linear system of dimensionless MHD equations we use the following anzats for the velocity \mathbf{V} , the magnetic field \mathbf{B} , the wave vector \mathbf{Q} , and the pressure P

$$\begin{aligned} \mathbf{V}(t, \mathbf{r}) &= \mathbf{V}_{\text{shear}}(\mathbf{r}) + \mathbf{V}_{\text{wave}}(t, \mathbf{r}), \\ \mathbf{V}_{\text{wave}} &= iV_A \sum_{\mathbf{Q}} \mathbf{v}_{\mathbf{Q}}(\tau) e^{i\mathbf{Q} \cdot \mathbf{x}}, \quad \mathbf{V}_{\text{shear}} = Ax \mathbf{e}_y, \end{aligned} \quad (1.13)$$

$$\mathbf{B}(t, \mathbf{r}) = \mathbf{B}_0 + \mathbf{B}_{\text{wave}}(t, \mathbf{r}), \quad \mathbf{B}_{\text{wave}}(t, \mathbf{r}) = B_0 \sum_{\mathbf{Q}} \mathbf{b}_{\mathbf{Q}}(\tau) e^{i\mathbf{Q} \cdot \mathbf{x}}, \quad (1.14)$$

$$P(t, \mathbf{r}) = P_0 + P_{\text{wave}}(t, \mathbf{r}), \quad P_{\text{wave}}(t, \mathbf{r}) = \rho V_A^2 \sum_{\mathbf{Q}} \mathcal{P}_{\mathbf{Q}}(\tau) e^{i\mathbf{Q} \cdot \mathbf{x}}. \quad (1.15)$$

where the sums are actually integrals with respect to 3-dimensional Eulerian wave-vector space with independent coordinates Q_x, Q_y, Q_z

$$\sum_{\mathbf{Q}} = \iiint_{-\infty, -\infty, -\infty}^{+\infty, +\infty, +\infty} \frac{dQ_x dQ_y dQ_z}{(2\pi)^3} = \int d\left(\frac{\mathbf{Q}}{2\pi}\right) = \int \frac{d^3 Q}{(2\pi)^3}, \quad (1.16)$$

i.e. the sum is a short notation for Fourier integration with omitted differentials, integral limits and 2π multipliers. For the static magnetic \mathbf{B}_0 field with magnitude $B_0 = \sqrt{B_{0y}^2 + B_{0z}^2}$ we suppose a vertical B_{0z} and an azimuthal B_{0y} components parameterized by an angle θ and an unit vector $\boldsymbol{\alpha}$. We also assume that the Alfvén velocity V_A is much smaller than the sound speed c_s

$$\mathbf{B}_0 = B_0 \boldsymbol{\alpha}, \quad \boldsymbol{\alpha} = (0, \alpha_y = \sin \theta, \alpha_z = \cos \theta), \quad \mathbf{V}_A = \frac{\mathbf{B}_0}{\sqrt{\mu_0 \rho}}, \quad V_A = \frac{B_0}{\sqrt{\mu_0 \rho}}.$$

In the equations above we used the dimensionless space-vector

$$\mathbf{X} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \equiv \frac{\mathbf{r}}{\Lambda} = \frac{A\mathbf{r}}{V_A}, \quad \Lambda \equiv \frac{V_A}{A}, \quad \mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad (1.17)$$

and its dimensionless wave-vector counterpart \mathbf{Q} . Λ is the characteristic length of the system which we suppose to be much smaller than space inhomogeneities, e.g. the accretion disk thickness.

1.2 Wave-vector representation

1.2.1 Linear terms

We have a space homogeneous physical system and indispensably its modes bear the character of plane waves. The purpose of the present section is to find the Fourier transformation of the all the terms in the MHD equations Eq. (1.13), Eq. (1.14) and Eq. (1.15).

Let us start, for example, with the pressure. According to Eq. (1.15) we have

$$\frac{\nabla P}{\rho} = \frac{iV_A^2}{\Lambda} \int P_{\mathbf{Q}}(\tau) \mathbf{Q} e^{i\mathbf{Q} \cdot \mathbf{x}} \frac{d^3 Q}{(2\pi)^3} = iAV_A \sum_{\mathbf{Q}} \mathbf{Q} P_{\mathbf{Q}}(\tau) e^{i\mathbf{Q} \cdot \mathbf{x}} \quad (1.18)$$

$$\hat{\mathcal{F}}\left(\frac{\nabla P}{\rho}\right) \equiv \int e^{-i\mathbf{Q} \cdot \mathbf{x}} \frac{\nabla P}{\rho} d^3 X = iAV_A \mathbf{Q} P_{\mathbf{Q}}(\tau). \quad (1.19)$$

Analogously, according to Eq. (1.13) and Eq. (1.14), for the partial time derivatives we obtain

$$\partial_t \mathbf{V} = iAV_A \sum_{\mathbf{Q}} e^{i\mathbf{Q} \cdot \mathbf{x}} \partial_\tau \mathbf{v}_{\mathbf{Q}}(\tau), \quad \hat{\mathcal{F}}(\partial_t \mathbf{V}) = iAV_A \partial_\tau \mathbf{v}_{\mathbf{Q}}(\tau), \quad (1.20)$$

$$\partial_t \mathbf{B} = AB_0 \sum_{\mathbf{Q}} e^{i\mathbf{Q} \cdot \mathbf{x}} \partial_\tau \mathbf{b}_{\mathbf{Q}}(\tau), \quad \hat{\mathcal{F}}(\partial_t \mathbf{B}) = AB_0 \partial_\tau \mathbf{b}_{\mathbf{Q}}(\tau). \quad (1.21)$$

More complicated are the Fourier transformations of the expressions, containing the shear flow $\mathbf{V}_{\text{shear}} = V_A X \mathbf{e}_y$ and the wave variables

$$\mathbf{V}_{\text{shear}} \cdot \nabla \mathbf{V}_{\text{wave}} = -AV_A \int e^{i\mathbf{Q} \cdot \mathbf{X}} X Q_y \mathbf{v}(\tau) \frac{d^3 Q}{(2\pi)^3}, \quad (1.22)$$

$$\mathbf{V}_{\text{shear}} \cdot \nabla \mathbf{B}_{\text{wave}} = iAB_0 \int e^{i\mathbf{Q} \cdot \mathbf{X}} X Q_y \mathbf{b}(\tau) \frac{d^3 Q}{(2\pi)^3}. \quad (1.23)$$

Let variables \mathbf{V}_{wave} or \mathbf{B}_{wave} be presented by their Fourier components $\psi(\mathbf{X}) = \sum_{\mathbf{Q}} e^{i\mathbf{Q} \cdot \mathbf{X}} \psi_{\mathbf{Q}}$ and $\psi_{\mathbf{Q}} = \hat{\mathcal{F}}(\psi(\mathbf{X}))$. Our task is to derive the Fourier transformation $\hat{\mathcal{F}}(\mathbf{X}\psi(\mathbf{X}))$. Using that $\mathbf{X}e^{i\mathbf{Q} \cdot \mathbf{X}} = -i\partial_{\mathbf{Q}}e^{i\mathbf{Q} \cdot \mathbf{X}}$ and the Gaussian theorem, $\int_{\mathcal{V}} d^3 Q \partial_{\mathbf{Q}} = \oint_{\partial\mathcal{V}} d\mathbf{S}$ applied for the whole volume \mathcal{V} in wave-vector space and its boundary $\partial\mathcal{V}$ we can make the partial integration

$$\begin{aligned} \mathbf{X}\psi(\mathbf{X}) &= \mathbf{X} \sum_{\mathbf{Q}} e^{i\mathbf{Q} \cdot \mathbf{X}} \psi_{\mathbf{Q}} \\ &= -i \sum_{\mathbf{Q}} \{(\partial_{\mathbf{Q}} e^{i\mathbf{Q} \cdot \mathbf{X}}) \psi_{\mathbf{Q}} = -e^{i\mathbf{Q} \cdot \mathbf{X}} \partial_{\mathbf{Q}} \psi_{\mathbf{Q}} + \partial_{\mathbf{Q}} [e^{i\mathbf{Q} \cdot \mathbf{X}} \partial_{\mathbf{Q}} \psi_{\mathbf{Q}}]\} = \sum_{\mathbf{Q}} e^{i\mathbf{Q} \cdot \mathbf{X}} i\partial_{\mathbf{Q}} \psi_{\mathbf{Q}}, \end{aligned} \quad (1.24)$$

in the limit

$$\lim_{Q \rightarrow \infty} (Q^3 \psi_{\mathbf{Q}}) = 0.$$

In such a way we derived the well-known in quantum mechanics operator representation $\hat{\mathbf{X}} = i\partial_{\mathbf{Q}}$ and derived the Fourier transformation

$$\hat{\mathcal{F}}(\mathbf{X}\psi(\mathbf{X})) = i\partial_{\mathbf{Q}} \psi_{\mathbf{Q}}. \quad (1.25)$$

This expression is analogous to the Fourier transformation of the ∇ -operator

$$\hat{\mathcal{F}}(\nabla_{\mathbf{X}}) = i\mathbf{Q}, \quad (1.26)$$

and gives

$$\hat{\mathcal{F}}(X \mathbf{e}_y \cdot \nabla_{\mathbf{X}}) = -Q_y \partial_{Q_x}, \quad \hat{\mathcal{F}}(\mathbf{V}_{\text{shear}} \cdot \nabla) = -AQ_y \partial_{Q_x}, \quad \mathbf{V}_{\text{shear}} = V_A X \mathbf{e}_y. \quad (1.27)$$

Those relations give that

$$\hat{\mathcal{F}}[D_t^{\text{shear}} \equiv \partial_t + \mathbf{V}_{\text{shear}} \cdot \nabla = \partial_t + AX \partial_Y] = A \{D_{\tau}^{\text{shear}} \equiv \partial_{\tau} - Q_y \partial_{Q_x} = \partial_{\tau} + \mathbf{U}_{\text{shear}}(\mathbf{Q}) \cdot \partial_{\mathbf{Q}}\}. \quad (1.28)$$

In other words Fourier transformation of a linearized substantial derivative is again a linearized substantial derivative, but only in the wave-vector space. For this purpose we introduced the field of shear flow in the wave-vector space $\mathbf{U}_{\text{shear}}(\mathbf{Q}) \equiv -Q_y \mathbf{e}_x$; confer this result with $\mathbf{V}_{\text{shear}}/V_A = X \mathbf{e}_y$. Returning back to the velocity and magnetic field we arrive at

$$\hat{\mathcal{F}}[(\partial_t + \mathbf{V}_{\text{shear}} \cdot \nabla) \mathbf{V}_{\text{wave}}] = iAV_A [\partial_{\tau} + \mathbf{U}_{\text{shear}}(\mathbf{Q}) \cdot \partial_{\mathbf{Q}}] \mathbf{v}_{\mathbf{Q}}, \quad (1.29)$$

$$\hat{\mathcal{F}}[(\partial_t + \mathbf{V}_{\text{shear}} \cdot \nabla) \mathbf{B}_{\text{wave}}] = AB_0 [\partial_{\tau} + \mathbf{U}_{\text{shear}}(\mathbf{Q}) \cdot \partial_{\mathbf{Q}}] \mathbf{b}_{\mathbf{Q}}. \quad (1.30)$$

For the derivation of these equations we used Eq. (1.27) and according Eq. (1.20) and Eq. (1.21) $\hat{\mathcal{F}}(\partial_t) = A\partial_\tau$.

Very simple is the Fourier transformation of the dissipative terms which is reduced to the properties of the Laplacian

$$\begin{aligned}
\nu_k \nabla^2 \mathbf{V} &= -\nu_k iV_A \int \mathbf{v}_Q(\tau) Q^2 e^{i\mathbf{Q}\cdot\mathbf{x}} \frac{d^3Q}{(2\pi)^3} = -\frac{iV_A}{\Lambda} \nu_k \int \mathbf{v}_Q(\tau) Q^2 e^{i\mathbf{Q}\cdot\mathbf{x}} \frac{d^3Q}{(2\pi)^3} \\
&= -iAV_A \nu'_k \int Q^2 \mathbf{v}_Q(\tau) e^{i\mathbf{Q}\cdot\mathbf{x}} \frac{d^3Q}{(2\pi)^3}, \\
\nu_m \nabla^2 \mathbf{B} &= -\frac{B_0}{\Lambda^2} \nu_m \int Q^2 \mathbf{b}_Q(\tau) \frac{d^3Q}{(2\pi)^3} = -AB_0 \nu'_m \int Q^2 \mathbf{b}_Q(\tau) \frac{d^3Q}{(2\pi)^3}, \\
\frac{1}{iAV_A} \hat{\mathcal{F}}(\nu \nabla^2 \mathbf{V}) &= -\nu'_{\text{kin}} Q^2 \mathbf{v}(\tau), \quad \frac{1}{AB_0} \hat{\mathcal{F}}(\nu_m \nabla^2 \mathbf{B}) = -\nu'_m Q^2 \mathbf{b}(\tau), \\
\nu'_k &\equiv \frac{A}{V_A^2} \nu_k, \quad \nu'_m \equiv \frac{A}{V_A^2} \nu_m.
\end{aligned} \tag{1.31}$$

Hereafter for all terms coming from the velocity equation Eq. (1.11) we will separate a factor iAV_A and for all terms from Eq. (1.12) we will separate a factor AB_0 . Those factors will be common for the final equations in the wave-vector space.

For Coriolis force density per unit mass $-2\boldsymbol{\Omega} \times \mathbf{V}$, we derive

$$\begin{aligned}
-2\boldsymbol{\Omega} \times \mathbf{V} &= -2A\omega(-V_y \mathbf{e}_x + V_x \mathbf{e}_y) \\
&= 2A\omega \begin{pmatrix} V_y \\ -V_x \\ 0 \end{pmatrix} = 2A\omega \begin{pmatrix} Ax + iV_A \int v_{y,Q}(\tau) e^{i\mathbf{Q}\cdot\mathbf{x}} \frac{d^3Q}{(2\pi)^3} \\ -iV_A \int v_{x,Q}(\tau) e^{i\mathbf{Q}\cdot\mathbf{x}} \frac{d^3Q}{(2\pi)^3} \\ 0 \end{pmatrix}, \\
\hat{\mathcal{F}}(-2\boldsymbol{\Omega} \times \mathbf{V}) &= iAV_A 2\omega(v_{y,Q}(\tau) \mathbf{e}_x - v_{x,Q}(\tau) \mathbf{e}_y).
\end{aligned} \tag{1.32}$$

The centrifugal term $2\omega A^2 x$ is irrelevant for the wave amplitude equations.

Furthermore, we calculate Lorentz force per unit mass $\mathbf{j} \times \mathbf{B}$, and using $B_0^2/\mu_0 = \rho V_A^2$ we have

$$\begin{aligned}
\left(\left(\frac{\nabla \times \mathbf{B}}{\mu_0} \right) \times \frac{\mathbf{B}_0}{\rho} \right) &= \frac{iB_0}{\mu_0 \rho \Lambda} \int (\mathbf{Q} \times \mathbf{b}_Q) \times (B_{0y} \mathbf{e}_y + B_{0z} \mathbf{e}_z) e^{i\mathbf{Q}\cdot\mathbf{x}} \frac{d^3Q}{(2\pi)^3} \\
&= iAV_A \int [(\mathbf{Q} \times \mathbf{b}_Q(\tau)) \times \boldsymbol{\alpha}] \frac{d^3Q}{(2\pi)^3}, \\
\hat{\mathcal{F}} \left(\left(\frac{\nabla \times \mathbf{B}}{\mu_0} \right) \times \frac{\mathbf{B}_0}{\rho} \right) &= iAV_A [(\mathbf{Q} \times \mathbf{b}_Q(\tau)) \times \boldsymbol{\alpha}].
\end{aligned} \tag{1.33}$$

We have also other two zero terms having no influence on the wave dynamics. From momentum equation Eq. (1.11) and from equation for magnetic field Eq. (1.12) we have

$$\mathbf{V}_{\text{shear}} \cdot \nabla \mathbf{V}_{\text{shear}} = Ax \mathbf{e}_y \cdot \nabla Ax \mathbf{e}_y = A^2 x \mathbf{e}_y \cdot \mathbf{e}_x \mathbf{e}_y = 0, \tag{1.34}$$

$$\mathbf{B}_0 \cdot \nabla \mathbf{V}_{\text{shear}} = AB_0(\alpha_y \mathbf{e}_y + \alpha_z \mathbf{e}_z) \cdot \mathbf{e}_x \mathbf{e}_y = 0. \tag{1.35}$$

For the last linear terms we have

$$\begin{aligned} \mathbf{V}_{\text{wave}} \cdot \nabla \mathbf{V}_{\text{shear}} &= iAV_A \int \mathbf{v}_{\mathbf{Q}}(\tau) \cdot \mathbf{e}_x \mathbf{e}_y e^{i\mathbf{Q} \cdot \mathbf{x}} \frac{d^3 Q}{(2\pi)^3}, \\ \hat{\mathcal{F}}(\mathbf{V}_{\text{wave}} \cdot \nabla \mathbf{V}_{\text{shear}}) &= iAV_A v_{x,\mathbf{Q}}(\tau) \mathbf{e}_y, \end{aligned} \quad (1.36)$$

$$\begin{aligned} \mathbf{B}_{\text{wave}} \cdot \nabla \mathbf{V}_{\text{shear}} &= AB_0 \int b_{x,\mathbf{Q}}(\tau) \mathbf{e}_y e^{i\mathbf{Q} \cdot \mathbf{x}} \frac{d^3 Q}{(2\pi)^3}, \\ \hat{\mathcal{F}}(\mathbf{B}_{\text{wave}} \cdot \nabla \mathbf{V}_{\text{shear}}) &= AB_0 b_{x,\mathbf{Q}}(\tau) \mathbf{e}_y, \end{aligned} \quad (1.37)$$

$$\begin{aligned} \mathbf{B}_0 \cdot \nabla \mathbf{V}_{\text{wave}} &= \frac{iV_A B_0}{\Lambda} \int (\boldsymbol{\alpha} \cdot i\mathbf{Q}) \mathbf{v}_{\mathbf{Q}}(\tau) e^{i\mathbf{Q} \cdot \mathbf{x}} \frac{d^3 Q}{(2\pi)^3}, \\ \hat{\mathcal{F}}(\mathbf{B}_0 \cdot \nabla \mathbf{V}_{\text{wave}}) &= -AB_0 (\boldsymbol{\alpha} \cdot \mathbf{Q}) \mathbf{v}_{\mathbf{Q}}(\tau). \end{aligned} \quad (1.38)$$

All linear terms are well-known from previous investigations of MHD waves in magnetized shear flows. In the next subsection we will derive the nonlinear terms describing the wave-wave interaction coming from the convective time derivative.

1.2.2 Nonlinear wave-wave interaction

In order to derive the nonlinear term in the momentum equation Eq. (1.11) we calculate $\mathbf{V}_{\text{wave}} \cdot \nabla \mathbf{V}_{\text{wave}}$ using $\nabla \mathbf{X} = \frac{A}{V_A} \mathbb{1} = \mathbb{1}/\Lambda$

$$\begin{aligned} \mathbf{V}_{\text{wave}} \cdot \nabla \mathbf{V}_{\text{wave}} &= \sum_{\mathbf{Q}'} iV_A \mathbf{v}(\tau, \mathbf{Q}') e^{i\mathbf{Q}' \cdot \mathbf{x}} \cdot \nabla \sum_{\mathbf{Q}''} iV_A \mathbf{v}(\tau, \mathbf{Q}'') e^{i\mathbf{Q}'' \cdot \mathbf{x}} = \\ &= -iAV_A \sum_{\mathbf{Q}'} \sum_{\mathbf{Q}''} \mathbf{v}_{\mathbf{Q}'} \cdot \mathbf{Q}'' \mathbf{v}_{\mathbf{Q}''} e^{i(\mathbf{Q}'+\mathbf{Q}'') \cdot \mathbf{x}}. \end{aligned} \quad (1.39)$$

For the sake of brevity in the last terms we will omit the time argument τ and write the wave-vector argument \mathbf{Q} as index. Making the Fourier transformation

$$\begin{aligned} \hat{\mathcal{F}}(\mathbf{V}_{\text{wave}} \cdot \nabla \mathbf{V}_{\text{wave}}) &\equiv \int d^3 X e^{-i\mathbf{Q} \cdot \mathbf{x}} (\mathbf{V}_{\text{wave}} \cdot \nabla \mathbf{V}_{\text{wave}}) \\ &= -iAV_A \sum_{\mathbf{Q}'} \sum_{\mathbf{Q}''} \mathbf{v}_{\mathbf{Q}'} \cdot \mathbf{Q}'' \mathbf{v}_{\mathbf{Q}''} \delta\left(\frac{\mathbf{Q}' + \mathbf{Q}'' - \mathbf{Q}}{2\pi}\right) = -iAV_A \sum_{\mathbf{Q}'} \mathbf{v}_{\mathbf{Q}-\mathbf{Q}'} \mathbf{v}_{\mathbf{Q}'} \cdot \mathbf{Q}. \end{aligned} \quad (1.40)$$

The velocity field

$$\mathbf{V}_{\text{wave}}(\tau, \mathbf{X}) = iV_A \int e^{\mathbf{Q} \cdot \mathbf{x}} \mathbf{v}(\tau, \mathbf{Q}) \frac{d^3 X}{(2\pi)^3}, \quad \mathbf{B}_{\text{wave}}(\tau, \mathbf{X}) = B_0 \int e^{\mathbf{Q} \cdot \mathbf{x}} \mathbf{b}(\tau, \mathbf{Q}) \frac{d^3 X}{(2\pi)^3}, \quad (1.41)$$

has to be real, hence the Fourier components should be odd for the velocity and even for the magnetic field

$$\mathbf{v}_{-\mathbf{Q}} = -\mathbf{v}_{\mathbf{Q}}, \quad \mathbf{b}_{-\mathbf{Q}} = \mathbf{b}_{\mathbf{Q}}. \quad (1.42)$$

Analogously, for the Fourier component of the wave-wave interaction part of the Lorentz force $\mathbf{j}_{\text{wave}} \times \mathbf{B}_{\text{wave}}$ we obtain

$$\begin{aligned} \hat{\mathcal{F}} \left(\frac{1}{\mu_0 \rho} (\nabla \times \mathbf{B}_{\text{wave}}) \times \mathbf{B}_{\text{wave}} \right) &= \int d^3 X e^{-i\mathbf{Q} \cdot \mathbf{X}} \left(\frac{1}{\mu_0 \rho} (\nabla \times \mathbf{B}_{\text{wave}}) \times \mathbf{B}_{\text{wave}} \right) \\ &= iAV_A \sum_{\mathbf{Q}'} (\mathbf{Q}' \times \mathbf{b}_{\mathbf{Q}'}) \times \mathbf{b}_{\mathbf{Q}-\mathbf{Q}'}. \end{aligned} \quad (1.43)$$

In such a way we derive the Fourier component of the nonlinear term of the momentum equation

$$\mathbf{N}_{v,\mathbf{Q}} \equiv \frac{1}{iAV_A} \hat{\mathcal{F}} \left(-\mathbf{V}_{\text{wave}} \cdot \nabla \mathbf{V}_{\text{wave}} + \frac{1}{\mu_0 \rho} (\nabla \times \mathbf{B}_{\text{wave}}) \times \mathbf{B}_{\text{wave}} \right) \quad (1.44)$$

$$= \sum_{\mathbf{Q}'} [\mathbf{v}_{\mathbf{Q}-\mathbf{Q}'} \mathbf{v}_{\mathbf{Q}'} \cdot \mathbf{Q} + (\mathbf{Q}' \times \mathbf{b}_{\mathbf{Q}'}) \times \mathbf{b}_{\mathbf{Q}-\mathbf{Q}'}]. \quad (1.45)$$

Analogously, for the other nonlinear terms $\mathbf{B}_{\text{wave}} \cdot \nabla \mathbf{V}_{\text{wave}}$ and $\mathbf{V}_{\text{wave}} \cdot \nabla \mathbf{B}_{\text{wave}}$ we have

$$\begin{aligned} \hat{\mathcal{F}} (\mathbf{B}_{\text{wave}} \cdot \nabla \mathbf{V}_{\text{wave}}) &= \int d^3 X e^{-i\mathbf{Q} \cdot \mathbf{X}} (\mathbf{B}_{\text{wave}} \cdot \nabla \mathbf{V}_{\text{wave}}) \\ &= -\frac{B_0 V_A}{\Lambda} \sum_{\mathbf{Q}'} \sum_{\mathbf{Q}''} \mathbf{b}_{\mathbf{Q}'} \cdot \mathbf{Q}'' \mathbf{v}_{\mathbf{Q}''} \delta \left(\frac{\mathbf{Q}' + \mathbf{Q}'' - \mathbf{Q}}{2\pi} \right) \\ &= -AB_0 \sum_{\mathbf{Q}'} \mathbf{b}_{\mathbf{Q}'} \cdot (\mathbf{Q} - \mathbf{Q}') \mathbf{v}_{\mathbf{Q}-\mathbf{Q}'}, \end{aligned} \quad (1.46)$$

$$\begin{aligned} \hat{\mathcal{F}} (\mathbf{V}_{\text{wave}} \cdot \nabla \mathbf{B}_{\text{wave}}) &= \int d^3 X e^{-i\mathbf{Q} \cdot \mathbf{X}} (\mathbf{V}_{\text{wave}} \cdot \nabla \mathbf{B}_{\text{wave}}) \\ &= -\frac{B_0 V_A}{\Lambda} \sum_{\mathbf{Q}'} \sum_{\mathbf{Q}''} \mathbf{v}_{\mathbf{Q}'} \cdot \mathbf{Q}'' \mathbf{b}_{\mathbf{Q}''} \delta \left(\frac{\mathbf{Q}' + \mathbf{Q}'' - \mathbf{Q}}{2\pi} \right) \\ &= -AB_0 \sum_{\mathbf{Q}'} \mathbf{v}_{\mathbf{Q}'} \cdot (\mathbf{Q} - \mathbf{Q}') \mathbf{b}_{\mathbf{Q}-\mathbf{Q}'}. \end{aligned} \quad (1.47)$$

Those terms participate in the equation for the magnetic field. For their difference we have

$$\begin{aligned} \mathbf{N}_{b,\mathbf{Q}} &\equiv \frac{1}{AB_0} \hat{\mathcal{F}} (\mathbf{B}_{\text{wave}} \cdot \nabla \mathbf{V}_{\text{wave}} - \mathbf{V}_{\text{wave}} \cdot \nabla \mathbf{B}_{\text{wave}}) \\ &= \sum_{\mathbf{Q}'} [\mathbf{v}_{\mathbf{Q}'} \cdot (\mathbf{Q} - \mathbf{Q}') \mathbf{b}_{\mathbf{Q}-\mathbf{Q}'} - \mathbf{b}_{\mathbf{Q}'} \cdot (\mathbf{Q} - \mathbf{Q}') \mathbf{v}_{\mathbf{Q}-\mathbf{Q}'}] \\ &= -\mathbf{Q} \times \sum_{\mathbf{Q}'} (\mathbf{v}_{\mathbf{Q}'} \times \mathbf{b}_{\mathbf{Q}-\mathbf{Q}'}) \end{aligned} \quad (1.48)$$

As the function in r-space

$$\text{rot} (\mathbf{V}_{\text{wave}} \times \mathbf{B}_{\text{wave}}) = \mathbf{B}_{\text{wave}} \cdot \nabla \mathbf{V}_{\text{wave}} - \mathbf{V}_{\text{wave}} \cdot \nabla \mathbf{B}_{\text{wave}} \quad (1.49)$$

has zero divergence

$$\text{div} [\text{rot} (\mathbf{V}_{\text{wave}} \times \mathbf{B}_{\text{wave}})] = 0 \quad (1.50)$$

its Fourier transform is transversal $\mathbf{Q} \cdot \mathbf{N}_{b,\mathbf{Q}} = 0$ and automatically $\mathbf{N}_{b,\mathbf{Q}}^\perp = \mathbf{N}_{b,\mathbf{Q}}$.

In order to merge the so derived nonlinear terms in the next subsection we will rederive the linear terms in Lagrangian wave-vector space.

1.3 Elimination of pressure in the final MHD equations

It is common in MHD to formally seek the limit of a particular expression for infinite sound speed $c_s \rightarrow \infty$. Due to the complexity of the problem this standard approach for consideration weak magnetic fields when $V_A \ll c_s$ is inapplicable in our problem and we have to look for direct elimination of the pressure. After substituting Fourier transformations in Eq. (1.11) and Eq. (1.12) we obtain

$$\begin{aligned} (\partial_\tau + \mathbf{U}_{\text{shear}} \cdot \partial_{\mathbf{Q}}) \mathbf{v}_{\mathbf{Q}} &= -v_{x,\mathbf{Q}} \mathbf{e}_y + \mathbf{Q} P_{\mathbf{Q}} + [(\mathbf{Q} \times \mathbf{b}_{\mathbf{Q}}) \times \boldsymbol{\alpha}] + 2\omega(v_{y,\mathbf{Q}} \mathbf{e}_x - v_{x,\mathbf{Q}} \mathbf{e}_y) \\ &\quad - \nu'_k Q^2 \mathbf{v}_{\mathbf{Q}} + \sum_{\mathbf{Q}'} [\mathbf{v}_{\mathbf{Q}-\mathbf{Q}'} \mathbf{v}_{\mathbf{Q}'} \cdot \mathbf{Q} + (\mathbf{Q}' \times \mathbf{b}_{\mathbf{Q}'} \times \mathbf{b}_{\mathbf{Q}-\mathbf{Q}'}], \\ (\partial_\tau + \mathbf{U}_{\text{shear}} \cdot \partial_{\mathbf{Q}}) \mathbf{b}_{\mathbf{Q}} &= b_{x,\mathbf{Q}} \mathbf{e}_y - (\mathbf{Q} \cdot \boldsymbol{\alpha}) \mathbf{v}_{\mathbf{Q}} - \nu'_m Q^2 \mathbf{b}_{\mathbf{Q}} - \mathbf{Q} \times \sum_{\mathbf{Q}'} (\mathbf{v}_{\mathbf{Q}'} \times \mathbf{b}_{\mathbf{Q}-\mathbf{Q}'}). \end{aligned}$$

For the sake of brevity we introduce

$$\begin{aligned} \mathcal{F}_{\mathbf{Q}} &\equiv Q_y \frac{\partial \mathbf{v}_{\mathbf{Q}}}{\partial Q_x} - \mathbf{e}_y \mathbf{e}_x \cdot \mathbf{v}_{\mathbf{Q}} + [(\mathbf{Q} \times \mathbf{b}) \times \boldsymbol{\alpha}] + 2\omega \times \mathbf{v}_{\mathbf{Q}} \\ &\quad + \nu'_k Q^2 \mathbf{v}_{\mathbf{Q}} + \sum_{\mathbf{Q}'} [\mathbf{v}_{\mathbf{Q}-\mathbf{Q}'} \mathbf{v}_{\mathbf{Q}'} \cdot \mathbf{Q} + (\mathbf{Q}' \times \mathbf{b}_{\mathbf{Q}'} \times \mathbf{b}_{\mathbf{Q}-\mathbf{Q}'}]. \end{aligned} \quad (1.51)$$

Then the equation for the velocity can be rewritten as

$$\partial_\tau \mathbf{v}_{\mathbf{Q}} = P_{\mathbf{Q}} \mathbf{Q} + \mathcal{F}_{\mathbf{Q}}. \quad (1.52)$$

In order to express the pressure, we multiply both sides of this equation by \mathbf{Q}

$$\partial_\tau (\mathbf{Q} \cdot \mathbf{v}_{\mathbf{Q}}) = Q^2 P_{\mathbf{Q}} + \mathbf{Q} \cdot \mathcal{F}_{\mathbf{Q}}. \quad (1.53)$$

The incompressibility condition $\mathbf{Q} \cdot \mathbf{v}_{\mathbf{Q}} = 0$ gives for the pressure the solution of the Poisson equation

$$\begin{aligned} \mathcal{P} = -\frac{\mathbf{Q} \cdot \mathcal{F}_{\mathbf{Q}}}{Q^2} &= -\frac{1}{Q^2} \{2\mathbf{Q} \cdot \mathbf{e}_y \mathbf{e}_x \cdot \mathbf{v}_{\mathbf{Q}} + 2\omega \times \mathbf{v}_{\mathbf{Q}} + \mathbf{Q} \cdot [(\mathbf{Q} \times \mathbf{b}_{\mathbf{Q}}) \times \boldsymbol{\alpha}]\} \\ &\quad - \frac{1}{Q^2} \sum_{\mathbf{Q}'} \{\mathbf{Q} \cdot \mathbf{v}_{\mathbf{Q}-\mathbf{Q}'} \mathbf{v}_{\mathbf{Q}'} \cdot \mathbf{Q} + [(\mathbf{Q}' \times \mathbf{b}_{\mathbf{Q}'} \times \mathbf{b}_{\mathbf{Q}-\mathbf{Q}'}] \cdot \mathbf{Q}\}, \end{aligned}$$

where we used the obvious vector relations

$$\begin{aligned} \mathbf{Q} \cdot \mathbf{e}_y \mathbf{e}_x \cdot \mathbf{v}_\mathbf{Q} &= v_x Q_y, & \mathbf{Q} \cdot (\boldsymbol{\omega} \times \mathbf{v}_\mathbf{Q}) &= \omega(Q_y v_x - Q_x v_y), \\ (\mathbf{Q} \times \mathbf{b}_\mathbf{Q}) \times \boldsymbol{\alpha} &= (\mathbf{Q} \cdot \boldsymbol{\alpha})(\mathbf{Q} \cdot \mathbf{b}) - Q^2(\mathbf{b} \cdot \boldsymbol{\alpha}). \end{aligned} \quad (1.54)$$

This formula for the pressure we substitute in the Eq. (1.52) which takes the form

$$\partial_\tau \mathbf{v}_\mathbf{Q} = \mathcal{F}_\mathbf{Q}^\perp = \mathcal{F}_\mathbf{Q} - \frac{\mathbf{Q} \otimes \mathbf{Q}}{Q^2} \mathcal{F}_\mathbf{Q} = \Pi^\perp \mathbf{Q} \mathcal{F}_\mathbf{Q}, \quad (1.55)$$

where

$$\Pi^\perp \mathbf{Q} \equiv \mathbb{1} - \frac{\mathbf{Q} \otimes \mathbf{Q}}{Q^2} = \mathbb{1} - \mathbf{n} \otimes \mathbf{n}, \quad \mathbf{n} \equiv \frac{\mathbf{Q}}{Q} \quad (1.56)$$

is the projection operator which applies to the part of a vector, perpendicular to the wave-vector. In other words the elimination of the pressure conserves the perpendicular part of the Fourier component of the force $\mathcal{F}_\mathbf{Q}$ in the used dimensionless variables. The equation Eq. (1.55) means that the velocity field remains orthogonal to the wave vector. If in the beginning $\mathbf{Q} \cdot \mathbf{v}_\mathbf{Q}(\tau_0) = 0$, the evolution gives that $\mathbf{Q} \cdot \mathbf{v}_\mathbf{Q}(\tau) = 0$ for every $\tau > \tau_0$.

Using that for the velocity as applicable for every orthogonal vector

$$\Pi^\perp \partial_\tau \mathbf{v}_\mathbf{Q} = \partial_\tau \mathbf{v}_\mathbf{Q}, \quad \Pi^\perp \mathbf{v}_\mathbf{Q} = \mathbf{v}_\mathbf{Q} \quad (1.57)$$

we can rewrite Eq. (1.55) as

$$\Pi^\perp (\partial_\tau \mathbf{v}_\mathbf{Q} - \mathcal{F}_\mathbf{Q}) = 0. \quad (1.58)$$

In order to take into account the $Q_y \frac{\partial \mathbf{v}_\mathbf{Q}}{\partial Q_x}$ term in Eq. (1.51) we use the obvious relations

$$Q_y \frac{\partial}{\partial Q_x} \left(\mathbf{v}_\mathbf{Q} \cdot \frac{\mathbf{Q} \mathbf{Q}}{Q^2} \right) = Q_y \frac{\partial \mathbf{v}_\mathbf{Q}}{\partial Q_x} \cdot \frac{\mathbf{Q} \mathbf{Q}}{Q^2} + \frac{Q_y v_{x,\mathbf{Q}}}{Q^2} \mathbf{Q} = 0, \quad \mathbf{v}_\mathbf{Q} \cdot \mathbf{Q} = 0. \quad (1.59)$$

Now we represent the projection of the advective term $\mathbf{U}_{\text{shear}} \cdot \partial_\mathbf{Q} \mathbf{v}_\mathbf{Q}$ as

$$\Pi^\perp \mathbf{Q} (\mathbf{U}_{\text{shear}} \cdot \partial_\mathbf{Q} \mathbf{v}_\mathbf{Q}) = -Q_y \frac{\partial \mathbf{v}_\mathbf{Q}}{\partial Q_x} - n_y \mathbf{n} v_{x,\mathbf{Q}} \quad (1.60)$$

and to arrive at the momentum equation in the form where the projection operator exists explicitly only in the nonlinear term

$$\begin{aligned} &(\partial_\tau + \mathbf{U}_{\text{shear}} \cdot \partial_\mathbf{Q}) \mathbf{v}_\mathbf{Q}(\tau) \\ &= -v_{x,\mathbf{Q}} \mathbf{e}_y + 2n_y \mathbf{n} v_{x,\mathbf{Q}} + 2\omega \mathbf{n} (n_y v_{x,\mathbf{Q}} - n_x v_{y,\mathbf{Q}}) + 2\boldsymbol{\omega} \times v_\mathbf{Q} + (\boldsymbol{\alpha} \cdot \mathbf{Q}) \mathbf{b}_\mathbf{Q} \\ &- \nu'_k Q^2 \mathbf{v}_\mathbf{Q} + \Pi^\perp \mathbf{Q} \sum_{\mathbf{Q}'} [\mathbf{v}_{\mathbf{Q}-\mathbf{Q}'} \mathbf{v}_{\mathbf{Q}'} \cdot \mathbf{Q} + (\mathbf{Q}' \times \mathbf{b}_{\mathbf{Q}'}) \times \mathbf{b}_{\mathbf{Q}-\mathbf{Q}'}], \end{aligned} \quad (1.61)$$

$$\begin{aligned} &(\partial_\tau + \mathbf{U}_{\text{shear}} \cdot \partial_\mathbf{Q}) \mathbf{b}_\mathbf{Q}(\tau) \\ &= b_{x,\mathbf{Q}} \mathbf{e}_y - (\mathbf{Q} \cdot \boldsymbol{\alpha}) \mathbf{v}_\mathbf{Q} - \nu'_m Q^2 \mathbf{b}_\mathbf{Q} - \mathbf{Q} \times \sum_{\mathbf{Q}'} (\mathbf{v}_{\mathbf{Q}'} \times \mathbf{b}_{\mathbf{Q}-\mathbf{Q}'}), \end{aligned} \quad (1.62)$$

$$\mathbf{v}_\mathbf{Q}(\tau_0) = \Pi^\perp \mathbf{v}_\mathbf{Q}(\tau_0), \quad \mathbf{b}_\mathbf{Q}(\tau_0) = \Pi^\perp \mathbf{b}_\mathbf{Q}(\tau_0). \quad (1.63)$$

For numerical calculations the incompressibility conditions $\mathbf{n} \cdot \mathbf{b}_Q = 0$ and $\mathbf{n} \cdot \mathbf{v}_Q = 0$ can be used as a criterion for the error.

Using the relation

$$[\mathbf{U}_{\text{shear}} \cdot \partial_Q \mathbf{b}_Q(\tau)] \cdot \mathbf{Q} = Q_y b_{x,Q}, \quad (1.64)$$

one can easily check that the equation for the evolution of the magnetic field Eq. (1.62) can also be presented as the evolution of its part, perpendicular to the wave-vector

$$\partial_\tau \mathbf{b}_Q + \Pi^\perp \mathbf{U}_{\text{shear}} \cdot \partial_Q \mathbf{b}_Q = \Pi^\perp \mathbf{e}_y \mathbf{e}_x \cdot \mathbf{b}_Q - (\mathbf{Q} \cdot \boldsymbol{\alpha}) \mathbf{v}_Q - \nu'_m Q^2 \mathbf{b}_Q - \mathbf{Q} \times \sum_{Q'} (\mathbf{v}_{Q'} \times \mathbf{b}_{Q-Q'}). \quad (1.65)$$

Together with $\mathbf{v}_Q = \Pi^\perp \mathbf{v}_Q$ this equation automatically gives $\mathbf{b}_Q = \Pi^\perp \mathbf{b}_Q$ and $\text{div} \mathbf{B} = 0$.

In the matrix form the system of MHD equations reads as

$$D_\tau \Psi = M \Psi + N, \quad (1.66)$$

where after some algebra [Biskamp (2003)]

$$\begin{aligned} N &= \begin{pmatrix} \mathbf{N}_{b,Q}^\perp \\ \mathbf{N}_{v,Q}^\perp \end{pmatrix} \\ &= \begin{pmatrix} -\mathbf{Q} \times \sum_{Q'} (\mathbf{v}_{Q'} \times \mathbf{b}_{Q-Q'}) \\ \Pi^\perp \mathbf{Q} \sum_{Q'} [\mathbf{v}_{Q-Q'} \mathbf{v}_{Q'} \cdot \mathbf{Q} + (\mathbf{Q}' \times \mathbf{b}_{Q'}) \times \mathbf{b}_{Q-Q'}] \end{pmatrix} \\ &= \begin{pmatrix} \sum_{Q'} (\mathbf{b}_{Q'} \mathbf{v}_{Q-Q'} - \mathbf{v}_{Q'} \mathbf{b}_{Q-Q'}) \cdot \mathbf{Q} \\ \Pi^\perp \mathbf{Q} \sum_{Q'} (\mathbf{v}_{Q'} \mathbf{v}_{Q-Q'} + \mathbf{b}_{Q'} \mathbf{b}_{Q-Q'}) \cdot \mathbf{Q} \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} M &= \left(\begin{array}{ccc|ccc} -\nu'_m Q^2 & 0 & 0 & -Q_\alpha & 0 & 0 \\ 1 & -\nu'_m Q^2 & 0 & 0 & -Q_\alpha & 0 \\ 0 & 0 & -\nu'_m Q^2 & 0 & 0 & -Q_\alpha \\ \hline Q_\alpha & 0 & 0 & 2n_y n_x (\omega + 1) - \nu'_k Q^2 & -2n_x n_x \omega + 2\omega & 0 \\ 0 & Q_\alpha & 0 & 2n_y n_y (\omega + 1) - (2\omega + 1) & -2n_x n_y \omega - \nu'_k Q^2 & 0 \\ 0 & 0 & Q_\alpha & 2n_y n_z (\omega + 1) & -2n_x n_z \omega & -\nu'_k Q^2 \end{array} \right), \\ \Psi_Q &= \begin{pmatrix} b_x \\ b_y \\ b_z \\ v_x \\ v_y \\ v_z \end{pmatrix} \end{aligned} \quad (1.67)$$

and $Q_\alpha \equiv \mathbf{Q} \cdot \boldsymbol{\alpha}$.

The matrix can also be represented as

$$M = \left(\begin{array}{c|c} M_{bb} & M_{bv} \\ \hline M_{vb} & M_{vv} \end{array} \right), \quad \Psi_Q = \begin{pmatrix} \mathbf{b} \\ \mathbf{v} \end{pmatrix}, \quad (1.68)$$

$$\mathbf{M}_{vv} = 2n_y \begin{pmatrix} n_x & 0 & 0 \\ n_y & 0 & 0 \\ n_z & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + 2\omega \begin{pmatrix} n_x n_y & (n_y^2 + n_z^2) & 0 \\ -(n_x^2 + n_z^2) & -n_x n_y & 0 \\ n_y n_z & -n_x n_z & 0 \end{pmatrix} - \nu'_k Q^2 \mathbb{1}, \quad (1.69)$$

$$\mathbf{M}_{vb} = Q_\alpha \mathbb{1}, \quad \mathbf{M}_{bv} = -Q_\alpha \mathbb{1}, \quad \mathbf{M}_{bb} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \nu'_m Q^2 \mathbb{1}. \quad (1.70)$$

With the help of the matrices in this representation in the next section we will make Lyapunov analysis of the linearized MHD equations.

1.4 Lyapunov analysis of the linearized system in Lagrangian variables

For small Q_y we may neglect the advective term $\mathbf{U}_{\text{shear}} \cdot \partial \mathbf{Q} = -Q_y \partial_{Q_x}$. Then the linearized MHD equations take the form

$$\mathbf{D}_\tau \Psi = \mathbf{M} \Psi, \quad (\mathbf{Q}, \mathbf{Q}) \cdot \Psi = 0. \quad (1.71)$$

To perform an instability analysis we make use of the exponential substitution $\Psi = \exp(\lambda \tau) \psi$, which leads to an eigenvalue problem with transversality conditions

$$\mathbf{M}(\mathbf{Q}) \Psi_{\mathbf{Q}} = \lambda \Psi_{\mathbf{Q}}, \quad \mathbf{Q} \cdot \mathbf{b}_{\mathbf{Q}} = 0 = \mathbf{Q} \cdot \mathbf{v}_{\mathbf{Q}}; \quad (1.72)$$

to make it short we can further omit the index \mathbf{Q}

$$(\mathbf{M} - \lambda \mathbb{1}) \Psi = 0, \quad \mathbf{Q} \cdot \mathbf{b} = 0 = \mathbf{Q} \cdot \mathbf{v}. \quad (1.73)$$

Should we substitute the incompressibility and transversality conditions

$$v_z = -\frac{Q_x v_x + Q_y v_y}{Q_z}, \quad b_z = -\frac{Q_x b_x + Q_y b_y}{Q_z}, \quad (1.74)$$

in the secular equation, we would end up with an overdetermined system. To avoid it, we omit the equations which initially have λb_z and λv_z terms, i.e. the 3-rd and the 6-th rows in the secular equation. In such a way we derive a secular equation for a reduced matrix

$$\begin{aligned} (\tilde{\mathbf{M}} - \lambda \mathbb{1}) \psi = 0, \quad \psi = \begin{pmatrix} b_x \\ b_y \\ v_x \\ v_y \end{pmatrix}, \quad (1.75) \\ \tilde{\mathbf{M}} = \left(\begin{array}{cc|cc} -\nu'_m Q^2 & 0 & -Q_\alpha & 0 \\ 1 & -\nu'_m Q^2 & 0 & -Q_\alpha \\ Q_\alpha & 0 & 2n_y n_x (\omega + 1) - \nu'_k Q^2 & -2n_x n_x \omega + 2\omega \\ 0 & Q_\alpha & 2n_y n_y (\omega + 1) - (2\omega + 1) & -2n_x n_y \omega - \nu'_k Q^2 \end{array} \right). \end{aligned}$$

This secular equation

$$P_4(\lambda; \mathbf{Q}, \nu'_m, \nu'_k) \equiv \det(\tilde{\mathbf{M}} - \lambda \mathbb{I}) = 0 \quad (1.76)$$

has 4 eigenvalues and via a calculation of the eigenvectors, we can derive b_z and v_z according to the transversality conditions Eq. (1.74).

For an ideal fluid $\nu'_m = 0 = \nu'_k$. Omitting the viscosity terms we have a relatively simple form for the secular equation

$$\begin{aligned} & P_4(\lambda; \mathbf{Q}, \nu'_m = 0, \nu'_k = 0) \\ &= \lambda^4 - 2n_y n_x \lambda^3 + \{[(4 - 8n_y^2)n_x^2 + 4 - 4n_y^2] \omega^2 + [(2 - 8n_y^2)n_x^2 - 4n_y^2 + 2] \omega + 2Q_\alpha^2\} \lambda^2 \\ &\quad - 2Q_\alpha^2 n_y n_x \lambda + 2Q_\alpha^2 (n_x^2 + 1) \omega + Q_\alpha^4 = 0. \end{aligned} \quad (1.77)$$

As we pointed out these eigenvalues give only a WKB approximation for the dynamics of MHD variables $\psi(\tau)$. For the special case of $Q_y = 0$, which corresponds to an axial-symmetric motion, with a rotation along the z-axis, the secular equation gives directly the increments of the linearized MHD equations.

$$\begin{aligned} & P_4(\lambda; Q_y = 0, \nu'_m = 0, \nu'_k = 0) \\ &= \lambda^4 + 2[Q_\alpha^2 + (1 + 2\omega)(n_x^2 + 1)\omega] \lambda^2 + 2Q_\alpha^2 (n_x^2 + 1) \omega + Q_\alpha^4 = 0. \end{aligned} \quad (1.78)$$

The most restricted case is for wave-vectors parallel to the rotation axis $\mathbf{Q} = Q \mathbf{e}_z$ when $Q_\alpha = Q_z \cos \theta$

$$\begin{aligned} & P_4(\lambda; Q_x = 0, Q_y = 0, \nu'_m = 0, \nu'_k = 0) \\ &= \lambda^4 + 2[Q_\alpha^2 + (1 + 2\omega)\omega] \lambda^2 + (Q_\alpha^2 + 2\omega) Q_\alpha^2 = 0. \end{aligned} \quad (1.79)$$

This is perhaps the most cited bi-quadratic equation in the whole history of science because it describes the magnetorotational instability (MRI) discovered by Velikhov [Velikhov (1959)] in 1959. In the astrophysics this equation was recognized and overexposed by many astrophysical grants 30 years later, see equation Eq. (111) of Ref. [Balbus and Hawley (1998)] and historical remarks therein. If we consider the special case of pure shear $\omega = 0$ with $Q_y = 0$ this dispersion equation gives the usual Alfvén waves

$$(\lambda^2 + Q_\alpha^2)^2 = 0, \quad \omega = |Q_\alpha|, \quad (1.80)$$

i.e. the rotation destabilizes the Alfvén waves. The polarization of the magnetic field and the velocity of the Alfvén waves are along the shear flow.

For pure axial magnetic field $\mathbf{B} = B \mathbf{e}_z$, i.e. $\alpha = (0, 0, 1)$, and $Q_\alpha = Q_z$. The matrix reduction is then given by simply erasing the z-components and taking into account only the x- and the y-projections of the equations of motions

$$\tilde{\mathbf{M}}_{\text{MRI}} = \left(\begin{array}{cc|cc} 0 & 0 & -Q_z & 0 \\ 1 & 0 & 0 & -Q_z \\ \hline Q_z & 0 & 0 & 2\omega \\ 0 & Q_z & -(2\omega + 1) & 0 \end{array} \right), \quad \psi = \begin{pmatrix} b_x \\ b_y \\ v_x \\ v_y \end{pmatrix}. \quad (1.81)$$

The secular equation is the equation for MRI Eq. (1.79) with $Q_\alpha = Q_z$.

The projection method can be generalized in the general case if we introduce 2 unit vectors perpendicular to the wave-vector $\mathbf{e}_Q = \mathbf{Q}/Q$

$$\begin{aligned} |2\rangle &= \mathbf{e}_2 = \frac{\mathbf{e}_z \times \mathbf{e}_Q}{|\mathbf{e}_z \times \mathbf{e}_Q|} = \frac{1}{\sqrt{Q_x^2 + Q_y^2}} \begin{pmatrix} -Q_y \\ Q_x \\ 0 \end{pmatrix}, \\ |1\rangle &= \mathbf{e}_1 = \frac{\mathbf{e}_2 \times \mathbf{e}_Q}{|\mathbf{e}_2 \times \mathbf{e}_Q|} = \frac{1}{\sqrt{Q_x^2 + Q_y^2} \sqrt{Q_x^2 + Q_y^2 + Q_z^2}} \begin{pmatrix} -Q_x Q_z \\ Q_y Q_z \\ -Q_y^2 - Q_x^2 \end{pmatrix}, \end{aligned}$$

and also the corresponding bra-vectors

$$\langle 1| = \frac{(-Q_x Q_z, Q_y Q_z, -Q_y^2 - Q_x^2)}{\sqrt{Q_x^2 + Q_y^2} \sqrt{Q_x^2 + Q_y^2 + Q_z^2}}, \quad (1.82)$$

$$\langle 2| = \frac{(-Q_y, Q_x, 0)}{\sqrt{Q_x^2 + Q_y^2}}. \quad (1.83)$$

For the degenerated case of $Q_x = 0 = Q_y$ we can regularize by choosing $Q_x = \iota$ and $Q_y = 0$. Then the limit $\iota \rightarrow 0$ gives the regularization $|1\rangle = \mathbf{e}_x$ and $|2\rangle = \mathbf{e}_y$. For all matrices $M_{\alpha,\beta}$ where $\alpha, \beta = b, v$ we calculate the matrix elements in the two-dimensional space

$$(\overline{M}_{\alpha,\beta})_{jj'} = \langle j| M_{\alpha,\beta} |j'\rangle, \quad \text{where } j, j' = 1, 2. \quad (1.84)$$

In such a way we obtain a reduced 4×4 matrix

$$\overline{M} = \left(\begin{array}{c|c} \overline{M}_{bb} & \overline{M}_{bv} \\ \hline \overline{M}_{vb} & \overline{M}_{vv} \end{array} \right) \quad (1.85)$$

whose eigenvectors are automatically perpendicular to \mathbf{Q} , simply because we have used the orthogonal to \mathbf{Q} space.

As a rule the linearized analysis is made in Lagrangian, moving, wave-vector space

$$d_\tau \mathbf{K}(\tau) = \mathbf{U}_{\text{shear}}(\mathbf{K}(\tau)), \quad (1.86)$$

with a time-dependent wave-vector

$$K_x = K_{x,0} - K_y(\tau - \tau_0), \quad K_y = \text{const}, \quad K_z = \text{const} \quad (1.87)$$

for each MHD wave.

In these coordinates for linearized waves the substantial time derivative $D_\tau^{\text{shear}} = d_\tau$ is reduced to a usual time derivative and the separation of variables gives a system of ordinary independent equations for every MHD wave

$$d_\tau \Psi_{\mathbf{K}}(\tau) = M(\mathbf{K}(\tau)) \Psi_{\mathbf{K}}(\tau), \quad \mathbf{K}(\tau) \cdot \mathbf{v}_{\mathbf{K}}(\tau) = 0, \quad \mathbf{K}(\tau) \cdot \mathbf{b}_{\mathbf{K}}(\tau) = 0. \quad (1.88)$$

In this linearized case it is possible to exclude b_z and v_z . In such a way we arrive at a simple-for-programming system of 4 equations

$$\begin{aligned} d_\tau \psi_{\mathbf{K}}(\tau) &= \tilde{\mathbf{M}}(\mathbf{K}(\tau)) \psi_{\mathbf{K}}(\tau), \\ b_z &= -(K_x(\tau)b_x + K_y b_y)/K_z, \quad v_z = -(K_x(\tau)v_x + K_y v_y)/K_z. \end{aligned} \quad (1.89)$$

For small K_y one can apply WKB approximation supposing exponential time dependence of the MHD variables $\Psi(\tau) \propto \exp(\lambda\tau)$ and the wave amplitudes. In the WKB approximation the energy amplification between $\tau = -\infty$ and $\tau = +\infty$ is given by the eigenvalue λ with the maximal real part

$$G \approx \exp \left(2 \int_{-\infty}^{\infty} d\tau \operatorname{Re} \lambda_{\max}(\mathbf{K}(\tau)) \right). \quad (1.90)$$

For the case of MRI with nonzero B_z the amplification factors are so giant that the linear analysis makes no sense because the nonlinear terms become rather important and we have a nonlinear saturation of the MRI. This saturation simulates strong turbulence for small wave-vectors, but definitely for large wave-vectors $|K_y| \gg 1$ at $\tau \rightarrow \infty$ we have a wave type turbulence with a given frequency.

We have to mention that the linearized case of pure shear is exactly integrable in terms of Heun functions [?, [Mishonov et al. \(2007\)](#)]. Investigating numerically this case with $\omega = 0$ and $B_z = 0$ in his PhD work [[Chagelishvili et al. \(1993\)](#)] T. Hristov discovered in 1990 the amplification of slow magnetosonic waves (SMW) by shear flows. Applied to the physics of accretion disks this amplification works even for purely azimuthal magnetic fields and gives a scenario for weak magnetic turbulence related to amplification of SMW. We had to wait 30 years of incubation period, cf. [[Dessler \(1970\)](#)], for the SMW amplification to be recognized as an important for the astrophysics phenomenon. In the next section we will consider how to proceed with the solution of MHD equations.

1.5 Energy density and power density

Our first step is to calculate the energy of plane MHD waves with time-dependent amplitudes. Using that

$$\int e^{i\mathbf{Q} \cdot \mathbf{X}} d^3 X = (2\pi)^3 \delta(\mathbf{Q}) \quad (1.91)$$

for the energy we obtain

$$\frac{1}{2} \int \left(\rho \mathbf{V}_{\text{wave}}^2 + \frac{1}{\mu_0} \mathbf{B}_{\text{wave}}^2 \right) d^3 X = \rho V_A^2 \sum_{\mathbf{Q}} \epsilon_{\mathbf{Q}}, \quad \epsilon_{\mathbf{Q}} \equiv \frac{1}{2} (\mathbf{v}_{\mathbf{Q}}^2 + \mathbf{b}_{\mathbf{Q}}^2), \quad (1.92)$$

i.e. the energy density is

$$\rho V_A^2 \iiint \frac{1}{2} [\mathbf{v}_{\mathbf{Q}}^2(\tau) + \mathbf{b}_{\mathbf{Q}}^2(\tau)] \frac{dQ_x dQ_y dQ_z}{(2\pi)^3}. \quad (1.93)$$

Analogously, with the help of the viscous stress tensor σ'_{ik} we express the volume density of the wave heating

$$\begin{aligned} Q_{\text{kin}}^{\text{wave}} &= \int \sigma'_{ik} \partial_k V_i^{\text{wave}} d^3x = \frac{1}{2} \int \sigma'_{ik} (\partial_k V_i^{\text{wave}} + \partial_i V_k^{\text{wave}}) d^3x \\ &= \frac{\eta}{2} \int (\partial_k V_i^{\text{wave}} + \partial_i V_k^{\text{wave}})^2 d^3x \\ &= \frac{\eta V_A^2}{2\Lambda^2} \int \left(\sum_{\mathbf{Q}} Q_i v_k e^{i\mathbf{Q} \cdot \mathbf{x}} + \sum_{\mathbf{Q}'} Q'_k v_i e^{i\mathbf{Q}' \cdot \mathbf{x}} \right)^2 d^3x = \rho A V_A^2 \nu'_k \sum_{\mathbf{Q}} Q^2 v_{\mathbf{Q}}^2. \end{aligned}$$

Similarly for the Ohmic part of the energy dissipation rate we have

$$Q_{\text{Ohm}}^{\text{wave}} = \mathbf{j} \cdot \mathbf{E} = \frac{1}{\mu_0^2 \sigma_{\text{Ohm}}} (\text{rot} \mathbf{B}_{\text{wave}})^2 = \frac{B_0^2}{\mu_0^2 \sigma_{\text{Ohm}}} \left(\sum_{\mathbf{Q}} \nabla \times \mathbf{b}_{\mathbf{Q}} e^{i\mathbf{Q} \cdot \mathbf{x}} \right)^2 = \rho A V_A^2 \nu'_m \sum_{\mathbf{Q}} Q^2 b_{\mathbf{Q}}^2. \quad (1.94)$$

The dissipation rate of a laminar shear flow is given according to Newton's formula

$$Q_{\text{kin}}^{\text{shear}} = \frac{\eta}{2} \int (\partial_k V_i^{\text{shear}} + \partial_i V_k^{\text{shear}})^2 d^3x = \frac{\eta}{2} A^2 (\delta_{k,x} \delta_{i,y} + \delta_{i,x} \delta_{k,y})^2 = \eta A^2. \quad (1.95)$$

Now we can calculate the total energy dissipation $Q_{\text{tot}} = Q_{\text{kin}}^{\text{shear}} + Q_{\text{kin}}^{\text{wave}} + Q_{\text{Ohm}}^{\text{wave}}$, the viscosity and the effective viscosity η_{eff}

$$\eta = \frac{Q_{\text{kin}}^{\text{shear}}}{A^2}, \quad \eta_{\text{eff}} = \rho \nu_{\text{eff}} = \frac{Q_{\text{tot}}}{A^2}. \quad (1.96)$$

In this way we can express the effective kinematic viscosity by the dimensionless Fourier components of the velocity and the magnetic field

$$\nu_{\text{eff}}(\tau) = \nu_k + \nu_k \sum_{\mathbf{Q}} Q^2 \mathbf{v}_{\mathbf{Q}}^2(\tau) + \nu_m \sum_{\mathbf{Q}} Q^2 \mathbf{b}_{\mathbf{Q}}^2(\tau). \quad (1.97)$$

For example, if we have static probability distribution functions for the velocity and the magnetic field, the enhancement factor of the effective viscosity is given by the time-averaged squares of the Fourier components for $\tau \gg 1$

$$\frac{\eta_{\text{eff}}}{\eta} = 1 + \sum_{\mathbf{Q}} Q^2 \langle \mathbf{v}_{\mathbf{Q}}^2 \rangle + \frac{\nu_m}{\nu_k} \sum_{\mathbf{Q}} Q^2 \langle \mathbf{b}_{\mathbf{Q}}^2 \rangle; \quad (1.98)$$

this important parameter determines the work of the accretion discs as a machine for making of compact astrophysical objects. The most simple scenario is to have the solution of the static

equations for the i -th iteration of $\Psi_{\mathbf{Q}}$ and to calculate the next $(i+1)$ -th iteration

$$\begin{aligned} \partial_{\bar{\tau}} \mathbf{v}_{\mathbf{Q}}^{(i+1)} = & -Q_y \frac{\partial \mathbf{v}_{\mathbf{Q}}^{(i+1)}}{\partial Q_x} = -v_{x,\mathbf{Q}}^{(i+1)} \mathbf{e}_y + 2 \frac{Q_y v_{x,\mathbf{Q}}^{(i+1)}}{Q^2} \mathbf{Q} \\ & + 2\omega \left[\mathbf{n} (n_y v_{x,\mathbf{Q}}^{(i+1)} - n_x v_{y,\mathbf{Q}}^{(i+1)}) + (v_{y,\mathbf{Q}}^{(i+1)} \mathbf{e}_x - v_{x,\mathbf{Q}}^{(i+1)} \mathbf{e}_y) \right] \\ & + (\boldsymbol{\alpha} \cdot \mathbf{Q}) \mathbf{b}_{\mathbf{Q}}^{(i+1)} - \nu'_k Q^2 \mathbf{v}_{\mathbf{Q}}^{(i+1)} + \Pi^\perp \mathbf{Q} \sum_{\mathbf{Q}'} \left[\mathbf{v}_{\mathbf{Q}'}^{(i)} \otimes \mathbf{v}_{\mathbf{Q}-\mathbf{Q}'}^{(i)} + \mathbf{b}_{\mathbf{Q}'}^{(i)} \otimes \mathbf{b}_{\mathbf{Q}-\mathbf{Q}'}^{(i)} \right] \cdot \mathbf{Q}, \end{aligned} \quad (1.99)$$

$$\begin{aligned} \partial_{\bar{\tau}} \mathbf{b}_{\mathbf{Q}}^{(i+1)} = & -Q_y \frac{\partial \mathbf{b}_{\mathbf{Q}}^{(i+1)}}{\partial Q_x} = b_{x,\mathbf{Q}}^{(i+1)} \mathbf{e}_y - (\mathbf{Q} \cdot \boldsymbol{\alpha}) \mathbf{v}_{\mathbf{Q}}^{(i+1)} - \nu'_m Q^2 \mathbf{b}_{\mathbf{Q}}^{(i+1)} \\ & + \sum_{\mathbf{Q}'} \left[\mathbf{b}_{\mathbf{Q}'}^{(i)} \otimes \mathbf{v}_{\mathbf{Q}-\mathbf{Q}'}^{(i)} - \mathbf{v}_{\mathbf{Q}'}^{(i)} \otimes \mathbf{b}_{\mathbf{Q}-\mathbf{Q}'}^{(i)} \right] \cdot \mathbf{Q}, \end{aligned} \quad (1.100)$$

$$\begin{aligned} \partial_{\bar{\tau}} \equiv & -Q_y \frac{\partial}{\partial Q_x}, \quad \bar{\tau} \equiv -\frac{Q_x}{Q_y}, \quad D_\tau = \partial_\tau + \partial_{\bar{\tau}}, \\ & \text{for independent variables } (\bar{\tau}, Q_y, Q_z), \quad Q_x = -Q_y \bar{\tau}. \end{aligned} \quad (1.101)$$

For cold protoplanetary disks the Ohmic resistivity of weakly ionized gas is very high and the effective viscosity is dominated in Eq. (1.98) by ν_m/ν_k term, in other words the viscosity of the protoplanetary disks is created by Ohmic dissipation. Completely opposite is the situation for the hot almost completely-ionized Hydrogen plasma in quasars. The Ohmic resistivity is negligible and the effective viscosity is created by the Fourier components of the MHD waves $\langle \mathbf{v}_{\mathbf{Q}}^2 \rangle$. Only for small wave-vectors the MHD turbulence remains strong turbulence. At large wave-vectors we have weak wave turbulence with wave-vectors going to infinity. In the next section we will consider the stability conditions which have to be checked.

1.6 Stability

The linear Lyapunov analysis which we outlined in Sec. 1.4 gives the idea what we have to do when we obtain the static solution $\Psi_{\mathbf{Q}}^{(0)} = (\mathbf{b}_{\mathbf{Q}}^{(0)}, \mathbf{v}_{\mathbf{Q}}^{(0)})$. In order to investigate the stability of this static solution we have to consider a small time-dependent deviation from this solution $\Psi_{\mathbf{Q}}^{(1)}(\tau) = (\mathbf{b}_{\mathbf{Q}}^{(1)}(\tau), \mathbf{v}_{\mathbf{Q}}^{(1)}(\tau))$. In this case, neglecting the quadratic terms with respect to $\Psi_{\mathbf{Q}}^{(1)}$, we find that the nonlinear terms in the MHD equations are linear integral operators in \mathbf{Q} -space

$$\begin{aligned} \hat{N}' \Psi_{\mathbf{Q}}^{(1)} = & \left(\sum_{\mathbf{Q}'} \left[\mathbf{Q} \cdot \mathbf{v}_{\mathbf{Q}'}^{(1)} \otimes \mathbf{v}_{\mathbf{Q}-\mathbf{Q}'}^{(0)} + \mathbf{Q} \cdot \mathbf{v}_{\mathbf{Q}'}^{(0)} \otimes \mathbf{v}_{\mathbf{Q}-\mathbf{Q}'}^{(1)} + (\mathbf{Q}' \times \mathbf{b}_{\mathbf{Q}'}^{(0)}) \times \mathbf{b}_{\mathbf{Q}-\mathbf{Q}'}^{(1)} + (\mathbf{Q}' \times \mathbf{b}_{\mathbf{Q}'}^{(1)}) \times \mathbf{b}_{\mathbf{Q}-\mathbf{Q}'}^{(0)} \right] \right. \\ & \left. - \mathbf{Q} \times \sum_{\mathbf{Q}'} (\mathbf{v}_{\mathbf{Q}'}^{(0)} \times \mathbf{b}_{\mathbf{Q}-\mathbf{Q}'}^{(1)} + \mathbf{v}_{\mathbf{Q}'}^{(1)} \times \mathbf{b}_{\mathbf{Q}-\mathbf{Q}'}^{(0)}) \right). \end{aligned}$$

We obtain new terms in the eigenvalue problem which finally is reduced to the problem of obtaining the maximal eigenvalue of an integral equation in which the coefficients are solutions of the static MHD equations. Now let us analyze the perspectives.

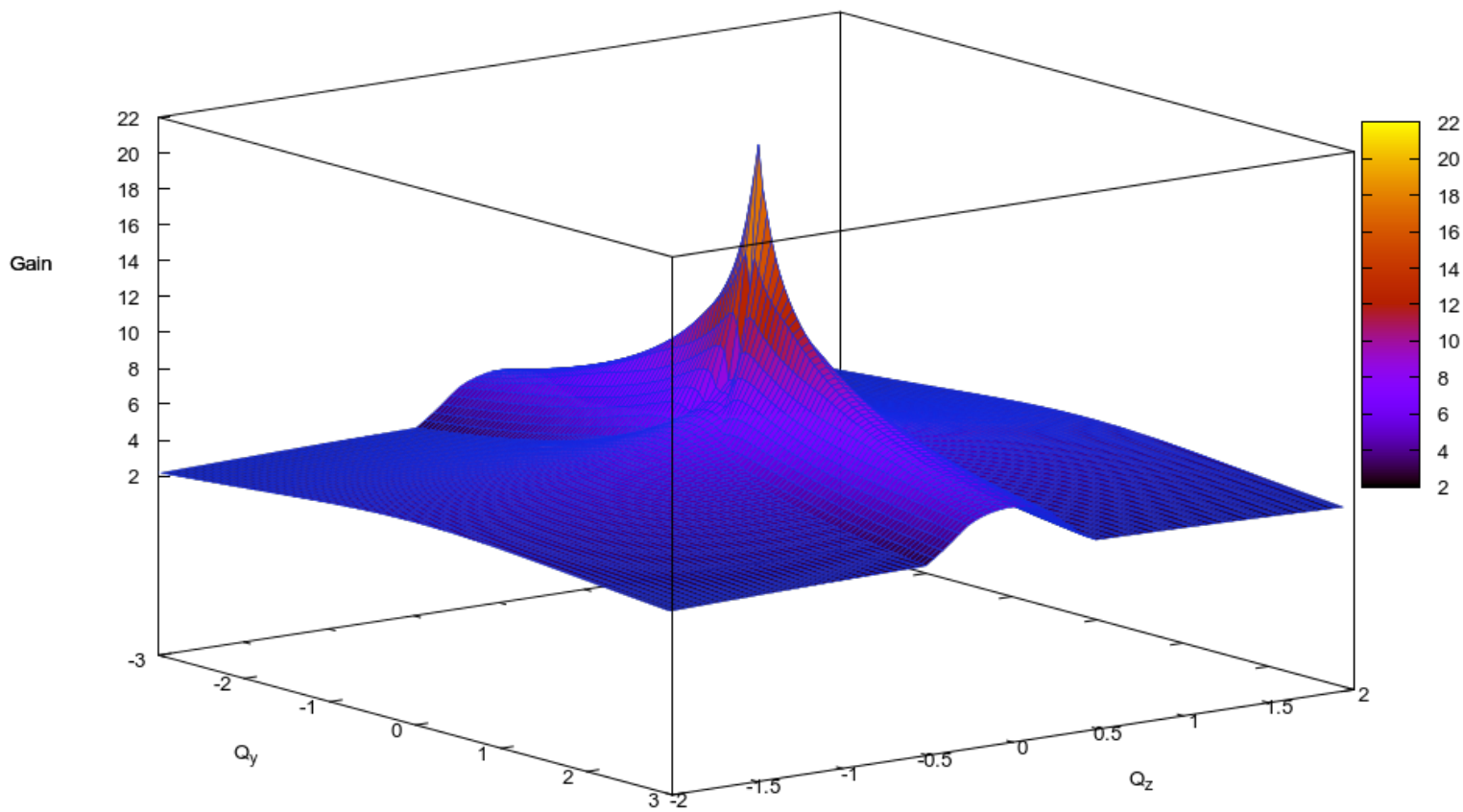


Figure 1.1: Gain for the case of $\omega = 0$, $\theta = 0$

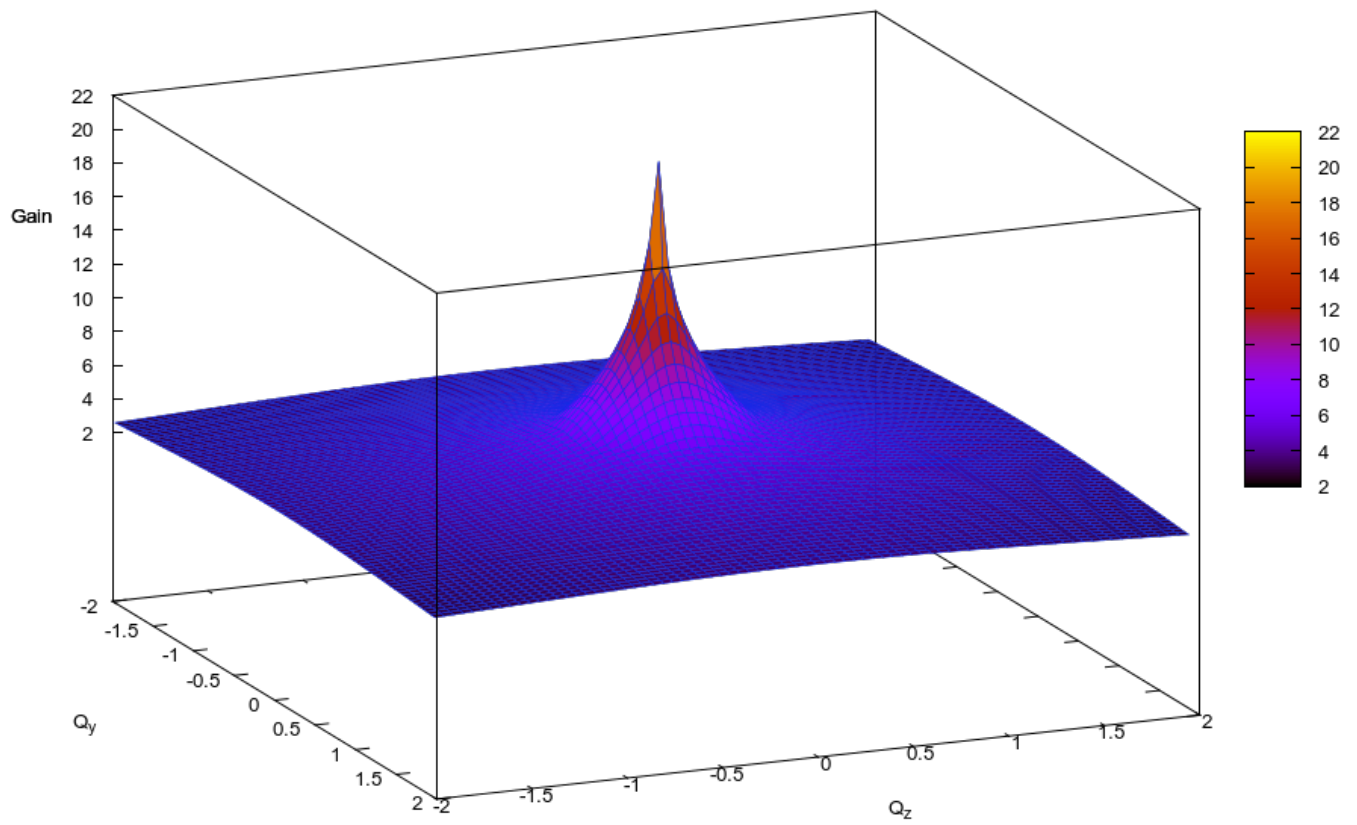


Figure 1.2: Gain for the case of $\omega = 0$, $\theta = \frac{\pi}{2}$

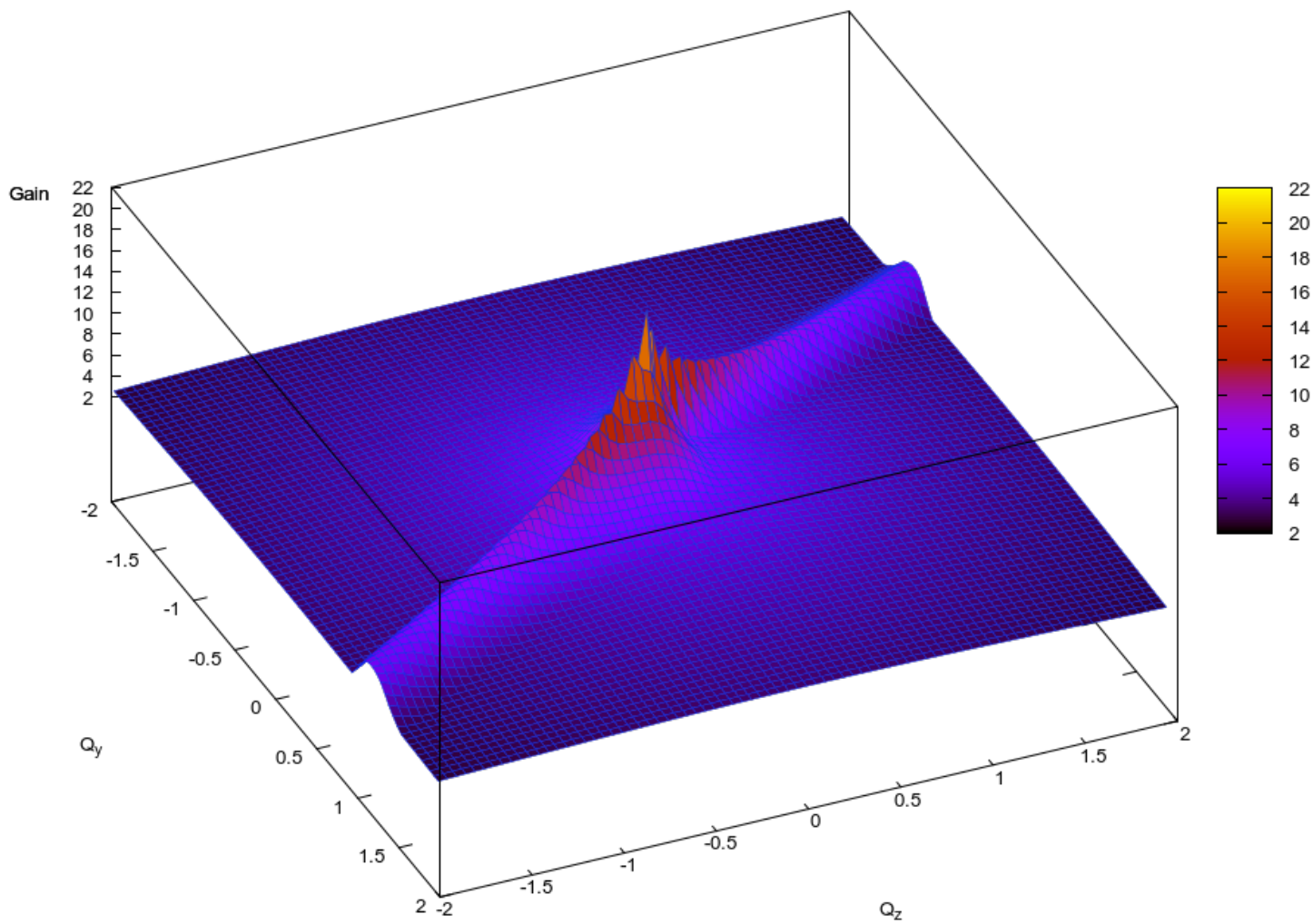


Figure 1.3: Gain for the case of $\omega = 0$, $\theta = \frac{\pi}{3}$

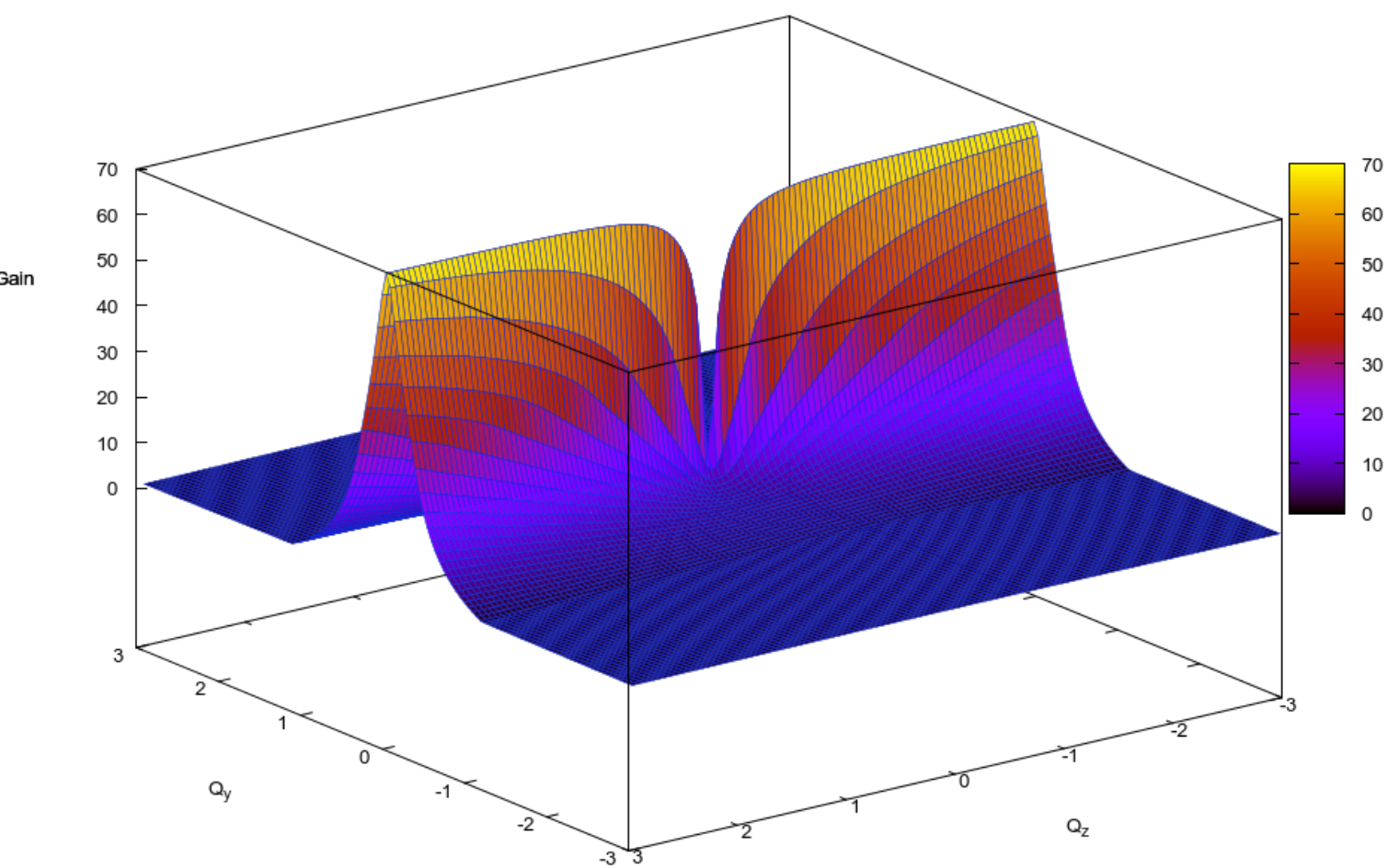


Figure 1.4: Gain for the case of $\omega = -\frac{2}{3}$, $\theta = \frac{\pi}{2}$

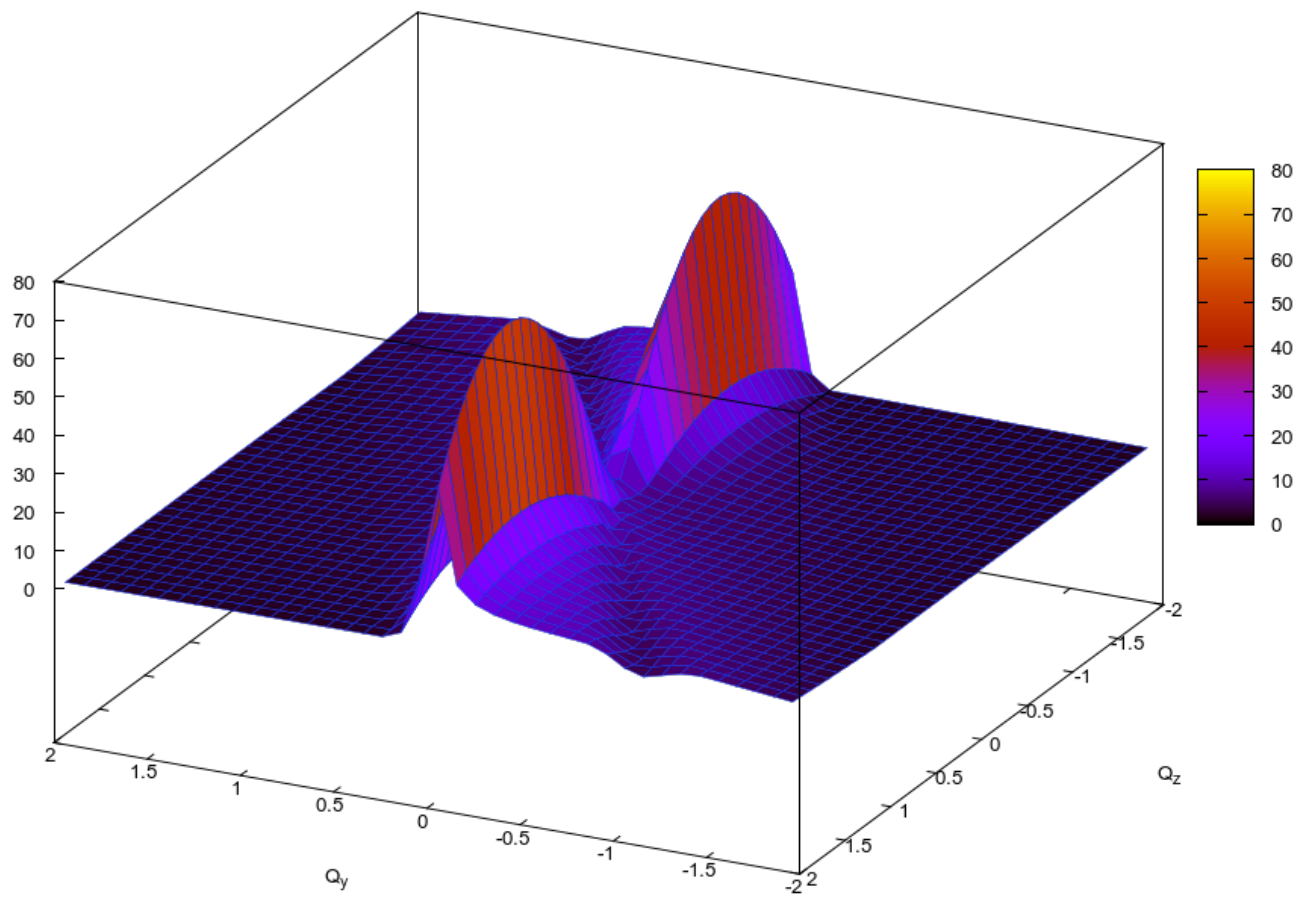


Figure 1.5: Gain for the case of $\omega = -\frac{2}{3}$, $\theta = \frac{\pi}{3}$

LINEAR ANALYSIS

2.1 Introduction

The aim of the present chapter is to derive the equations which have to be numerically solved in order to calculate the many orders of magnitude enhancement of the effective viscosity. We generalize the magnetohydrodynamic equations used in the original PhD work by Hristov [Chagelishvili et al.(1993)] and arrive to the conclusion that divergence SMW amplification coefficient is the mechanism which trigger the self-sustained turbulence in accretion disks. As the problem is a statistical one for short wavelength SMW our starting point in the next section 2.2 is the MHD equations in rectangular Cartesian coordinates. In this section are given the main MHD equations Eq. (2.6) and Eq. (2.7) in the wave-vector space. Linearized equations considered in sec. 2.3 is the first step of the analysis of the derived equations. The phase portraits considered in sec. 2.4 is the important tool for our understanding of SMW amplification – the heart of the disk turbulence, cf. page 28 [Balbus and Hawley (1998)]. Test for the accuracy of the numerical calculation is the analytical solution of SMW amplification for the case of pure shear described in sec. 2.5. The well-known magnetorotational instability (MRI) and the corresponding deviation equations are rederived in sec. 2.7 as special case of linearized MHD equations for SMW. It is considered in the final section 2.9 that the derived equations open the perspective for creation of self-sustained wave turbulence for the interesting for astrophysics case of large Reynolds numbers.

2.2 Model and description of the basic equations

First of all let us clarify that for astrophysical applications we suppose hydrogen plasma. For completely ionized plasma we have well known expression for the magnetic diffusivity ν_m and

kinematic viscosity ν_k

$$\nu_m = \varepsilon_0 c^2 \varrho = \frac{e^2 c^2 m_e^{1/2} \mathcal{L}_e}{0.6 \times 4\pi T_e^{3/2}} \ll \nu_k = \frac{\eta}{\rho} = \frac{0.4 T_p^{5/2}}{e^4 n_p M_p^{1/2} \mathcal{L}_p}, \quad \rho = n_p m_p, \quad \varepsilon_0 = \frac{1}{4\pi} \quad (2.1)$$

$$\mathcal{L}_p = \ln\left(\frac{\lambda_D T_p}{e^2}\right), \quad \mathcal{L}_e = \ln\left(\frac{\lambda_D T_e}{e^2}\right), \quad \frac{1}{\lambda_D^2} = 4\pi e^2 \left(\frac{n_e}{T_e} + \frac{n_p}{T_p}\right), \quad e^2 = \frac{q_e^2}{4\pi \varepsilon_0}. \quad (2.2)$$

At high temperatures the magnetic Prandtl number is $P_m \equiv \nu_k/\nu_m \propto T^4 \gg 1$. Here we use the self-explaining notations for the hydrogen plasma: the masses of electron m_e and proton M_p , the number of electrons n_e and n_p protons per unit volume, the density ρ and the viscosity η , electron T_e and proton temperature T_p , Coulomb logarithms \mathcal{L}_e and \mathcal{L}_p , Debye screening length λ_D , and the coefficient in Coulomb interaction e^2 . We will suppose equal temperatures $T_e = T_p$ and of charge neutrality $n_p = n_e$. The bare viscosity is determined by momentum transport of the elementary particles of the fluid: molecules, atoms, ions, protons and electrons is known as molecular viscosity.

For weak magnetic fields the Alfvén velocity V_A is much smaller than the sound c_s speed

$$V_A = \frac{B_0}{\sqrt{\mu_0 \rho}} \ll c_s = \sqrt{\gamma \frac{P}{\rho}}, \quad \gamma \equiv \frac{c_p}{c_v} = \frac{5}{3}. \quad (2.3)$$

This strong inequality for weak magnetic fields lead that for slow magnetosonic waves (SMW) we can use the incompressible fluid $\text{div} \mathbf{V} = 0$ approximation. In this approximation the slow magnetosonic waves have the same dispersion as Alfvén waves (AW) and are often called pseudo-Alfvén waves.

We suppose the geometry of an accretion disk and choose the z -axis along the axis of rotation with angular velocity Ω . The y -axis we choose parallel to the shear flow which is local \mathbf{e}_φ direction. The local x -axis is chosen to be along the local r direction. The modulus of the shear velocity depends on distance to the central compact object r . In the so chosen coordinates for the shear flow we have

$$\mathbf{V}_{\text{shear}} = A x \mathbf{e}_y, \quad \mathbf{e}_x = \mathbf{e}_r, \quad \mathbf{e}_y = \mathbf{e}_\varphi, \quad (2.4)$$

where A is the shear rate with dimension of frequency.

Introducing the field of shear flow in wave-vector space (“momentum space” in the quantum mechanics) and substantial Lagrangian derivative

$$\mathbf{U}_{\text{shear}}(\mathbf{Q}) \equiv -Q_y \mathbf{e}_x, \quad D_\tau^{\text{shear}} \equiv \partial_\tau + \mathbf{U}_{\text{shear}}(\mathbf{Q}) \cdot \partial_{\mathbf{Q}} = \partial_\tau - Q_y \partial_{Q_x} \quad (2.5)$$

the MHD equations in wave-vector space read [[Chagelishvili et al.\(1993\)](#), [Dimitrov et al. \(2011\)](#)]

as

$$\begin{aligned} D_\tau^{\text{shear}} \mathbf{v}_\mathbf{Q}(\tau) = & -v_{x,\mathbf{Q}} \mathbf{e}_y + 2n_y \mathbf{n} v_{x,\mathbf{Q}} + 2\omega_c \mathbf{n} (n_y v_{x,\mathbf{Q}} - n_x v_{y,\mathbf{Q}}) + 2\omega_c \times \mathbf{v}_\mathbf{Q} + (\boldsymbol{\alpha} \cdot \mathbf{Q}) \mathbf{b}_\mathbf{Q} \\ & - \nu'_k Q^2 \mathbf{v}_\mathbf{Q} + \Pi^{\perp \mathbf{Q}} \cdot \sum_{\mathbf{Q}'} [\mathbf{v}_{\mathbf{Q}'} \otimes \mathbf{v}_{\mathbf{Q}-\mathbf{Q}'} + \mathbf{b}_{\mathbf{Q}'} \otimes \mathbf{b}_{\mathbf{Q}-\mathbf{Q}'}] \cdot \mathbf{Q}, \end{aligned} \quad (2.6)$$

$$\begin{aligned} D_\tau^{\text{shear}} \mathbf{b}_\mathbf{Q}(\tau) = & b_{x,\mathbf{Q}} \mathbf{e}_y - (\mathbf{Q} \cdot \boldsymbol{\alpha}) \mathbf{v}_\mathbf{Q} - \nu'_m Q^2 \mathbf{b}_\mathbf{Q} \\ & + \Pi^{\perp \mathbf{Q}} \cdot \sum_{\mathbf{Q}'} [\mathbf{b}_{\mathbf{Q}'} \otimes \mathbf{v}_{\mathbf{Q}-\mathbf{Q}'} - \mathbf{v}_{\mathbf{Q}'} \otimes \mathbf{b}_{\mathbf{Q}-\mathbf{Q}'}] \cdot \mathbf{Q}, \end{aligned} \quad (2.7)$$

where velocity and magnetic field are perpendicular to the wave-vector

$$\mathbf{v}_\mathbf{Q}(\tau_0) = \Pi^{\perp} \mathbf{v}_\mathbf{Q}(\tau_0), \quad \mathbf{b}_\mathbf{Q}(\tau_0) = \Pi^{\perp} \mathbf{b}_\mathbf{Q}(\tau_0), \quad \Pi^{\perp \mathbf{Q}} \equiv \mathbb{1} - \frac{\mathbf{Q} \otimes \mathbf{Q}}{Q^2} = \mathbb{1} - \mathbf{n} \otimes \mathbf{n}, \quad \mathbf{n} \equiv \frac{\mathbf{Q}}{Q}, \quad (2.8)$$

and the $\Pi^{\perp \mathbf{Q}}$ is the projection operator.

In the next section we will analyze the linear terms known from other works on space plasmas.

2.3 Linear terms

The linearized MHD equations in many special cases of magnetized shear flows are investigated in great details. One of the purposes of the present section is to give different tests for the validity of the general equations which was written in Eulerian time independent wave-vector space \mathbf{Q} .

If the nonlinear terms are negligible the linearized equations is better to be analyzed in Lagrangian description in wave-vector space. Let us introduce time-dependent wave-vector

$$d_\tau \mathbf{K}(\tau) = \mathbf{U}_{\text{shear}}(\mathbf{K}(\tau)). \quad (2.9)$$

This equation according Eq. (2.5) has the solution

$$\begin{aligned} K_z = \text{const}, \quad K_y = \text{const}, \quad K_x = K_{x,0} - K_y \tau, \quad K_{x,0} = \text{const}, \\ K_\alpha \equiv \boldsymbol{\alpha} \cdot \mathbf{K} = K_y \sin \theta + K_z \cos \theta. \end{aligned} \quad (2.10)$$

For $K_y \neq 0$ we can parameterize $\tau_0 = K_{x,0}/K_y$ then we have $K_x(\tau) = -(\tau - \tau_0)K_y$. Considering only one wave without restrictions we can choose $\tau_0 = 0$.

Now Lagrangian derivative is converted to ordinary time derivative

$$D_\tau^{\text{shear}} \rightarrow d_\tau, \quad (2.11)$$

and in the MHD equations we have formally to replace the index \mathbf{Q} by \mathbf{K} , i.e.

$$\mathbf{v}_\mathbf{Q} \rightarrow \mathbf{v}_\mathbf{K} \rightarrow \mathbf{v}, \quad \mathbf{b}_\mathbf{Q} \rightarrow \mathbf{b}_\mathbf{K} \rightarrow \mathbf{b}, \quad \mathbf{Q} \rightarrow \mathbf{K}. \quad (2.12)$$

After this substitution the linearized system Eq. (2.6) and Eq. (2.7) takes the form

$$d_\tau \mathbf{v} = -v_x \mathbf{e}_y + 2n_y \mathbf{n} v_x + K_\alpha \mathbf{b} - \nu'_k K^2 \mathbf{v} + 2\omega_c [\mathbf{n} (n_y v_x - n_x v_y) + (v_y \mathbf{e}_x - v_x \mathbf{e}_y)] \quad (2.13)$$

$$d_\tau \mathbf{b} = b_x \mathbf{e}_y - K_\alpha \mathbf{v} - \nu'_m K^2 \mathbf{b}, \quad \mathbf{K} \cdot \mathbf{b} = 0, \quad \mathbf{K} \cdot \mathbf{v} = 0, \quad (2.14)$$

and the expression for the pressure taking into account Eq. (2.14) reads as

$$\mathcal{P} = -\frac{1}{K^2} \{2K_y v_x + (\mathbf{b} \cdot \boldsymbol{\alpha}) K^2 + 2\omega_c (K_y v_x - K_x v_y)\}. \quad (2.15)$$

The simplest derivation of the linearized system Eq. (2.13), Eq. (2.14) and Eq. (2.15) is to substitute the single wave ansatz

$$\mathbf{V}(t, \mathbf{r}) = Ax\mathbf{e}_y + iV_A \mathbf{v}(\tau) e^{i\mathbf{k} \cdot \mathbf{r}}, \quad v \ll 1, \quad (2.16)$$

$$\mathbf{B}(t, \mathbf{r}) = B_0 \boldsymbol{\alpha} + B_0 \mathbf{b}(\tau) e^{i\mathbf{k} \cdot \mathbf{r}}, \quad b \ll 1, \quad (2.17)$$

$$\mathbf{k}(t) = -k_y \tau \mathbf{e}_x + k_y \mathbf{e}_y + k_z \mathbf{e}_z, \quad \tau \equiv At. \quad (2.18)$$

into MHD equations Eq. (1.11) and Eq. (1.12) in coordinate space and to eliminate the pressure.

In the next section we will analyze the phase the amplification of SMW in Lagrangian variables.

2.4 Phase portraits of amplification of SMW

On Fig. 2.1 are shown numerical solutions of the system of equations Eq. (2.13) and Eq. (2.14) corresponding to initial conditions:

$$\begin{aligned} v_x &= 0.0003162, \quad v_y = -0.033329, \quad v_z = 0.0000008, \quad K_y = 0.17, \\ b_x &= 0.0099995, \quad b_y = -1.054045, \quad b_z = 0.0000158, \quad K_z = 0.1. \end{aligned} \quad (2.19)$$

For the radial motion the velocity and the magnetic field start at $\xi = -\infty$ and finish at $\xi = \infty$ with zero values. Only at the moment of wave amplification $\xi = 0$ and $K_x = 0$ we have significant radial components (v_x, b_x) . For azimuthal motion (v_y, b_y) , which describes the SMW, we assume an initial amplitude, which is amplified by shear flow. This is the main feature of the phenomenon which we explore. Axial components (v_z, b_z) , which describe AW, has no amplification; at $\xi = -\infty$ they have zero value. Nonzero area of asymptotic cycle in the plane (v_z, b_z) corresponds to the conversion of SMW in the AW. The ratio of dimensionless wave energy density $w = \frac{1}{2}(\mathbf{v}^2 + \mathbf{b}^2)$ at $\xi = \infty$ and $\xi = -\infty$ describes the wave gain $G = w(\infty)/w(-\infty)$.

In order to check the validity of the linearized equations in the next section we present the solvable case without rotation.

2.5 Analytical solution for pure shear of an inviscid magnetized fluid

For non-rotating $\omega_c = 0$ ideal $\nu'_m = 0 = \nu'_k$ fluid is the MHD system Eq. (2.13) and Eq. (2.14) reduced to two second order equations. This result was initially [?] derived only for toroidal static magnetic fields $B_{0,z} = 0$. Introducing new variables

$$K_\perp \equiv \sqrt{K_y^2 + K_z^2}, \quad \tilde{K} \equiv \frac{K_\perp}{K_y} K_\alpha, \quad \xi \equiv \frac{K_y}{K_\perp} \tau, \quad \psi(\xi) \equiv b_x(\xi) \sqrt{1 + \xi^2}, \quad (2.20)$$

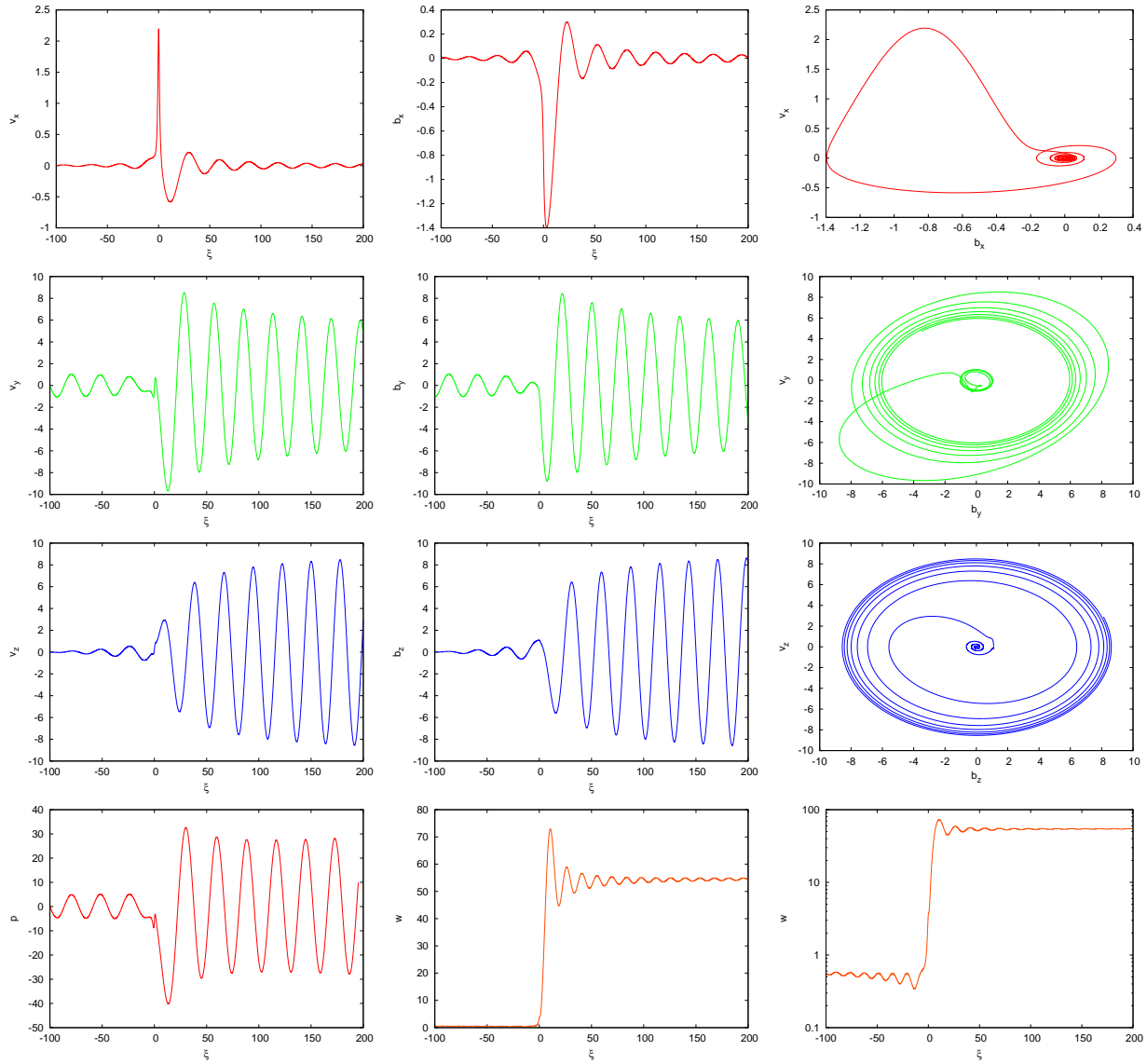


Figure 2.1: Velocity $\mathbf{v}(\xi)$, magnetic field $\mathbf{b}(\xi)$, pressure p , total wave energy w and phase portraits of radial $e_r = e_x$ (upper row), azimuthal $e_\varphi = e_y$ (middle row) and axial $e_z = e_z$ (bottom row) components of the fields. The area of the asymptotic cycles is proportional to the energy of each component. The increasing of the area of azimuthal component (v_y , b_y) describes the amplification of SMW – the main effect investigated in the present work. The phase portraits of the right hand side trivialize the analytically derived amplification of SMW. For different projections we have transmissions from zero or finite values of the asymptotic area describing energy distribution of the wave.

we obtain Schrödinger type equation with solution expressed by confluent Heun functions

$$d_\xi^2 \psi + \left[\tilde{K}^2 - \frac{1}{(1 + \xi^2)^2} \right] \psi = 0, \quad \psi(\xi) = C_g \psi_g(\xi) + C_u \psi_u(\xi), \quad (2.21)$$

$$\psi_g = \sqrt{1 + \xi^2} \text{HeunC}(0, -\frac{1}{2}, 0, -\frac{\tilde{K}^2}{4}, \frac{1 + \tilde{K}^2}{4}, -\xi^2), \quad (2.22)$$

$$\psi_u = \xi \sqrt{1 + \xi^2} \text{HeunC}(0, +\frac{1}{2}, 0, -\frac{\tilde{K}^2}{4}, \frac{1 + \tilde{K}^2}{4}, -\xi^2), \quad (2.23)$$

where C_g and C_u are arbitrary constants. Formal correspondence to quantum mechanics is by replacements

$$\frac{2m}{\hbar^2} U(\xi) \rightarrow \frac{1}{(1 + \xi^2)^2}, \quad \frac{2m}{\hbar^2} E \rightarrow \tilde{K}^2. \quad (2.24)$$

For the z -component of the magnetic field we have equation similar to forced harmonic oscillator

$$(d_\xi^2 + \tilde{K}^2) b_z = 2K_z \frac{v_x}{1 + \xi^2}. \quad (2.25)$$

In such a way using also the condition $\mathbf{K} \cdot \mathbf{b} = 0$ we obtain the analytical solution for the magnetic field. The dimensionless magnetic field \mathbf{b} expressed by the effective ψ -function

$$\begin{aligned} b_x &= \frac{\psi(\xi)}{\sqrt{1 + \xi^2}}, \quad \chi(\xi) \equiv \tilde{C}_g \cos(L\xi) + \tilde{C}_u \frac{\sin(\tilde{K}\xi)}{\tilde{K}}, \\ b_y &= -\frac{2K_z^2}{K_y \tilde{K}} \int_{-\infty}^{\xi} \sin[\tilde{K}(\xi - \xi')] \frac{v_x(\xi')}{1 + \xi'^2} d\xi' + \frac{\tilde{K}}{K_y} \frac{\xi \psi(\xi)}{\sqrt{1 + \xi^2}} - \frac{K_z}{K_y} \chi(\xi), \\ b_z &= \frac{2K_z}{\tilde{K}} \int_{-\infty}^{\xi} \sin[\tilde{K}(\xi - \xi')] \frac{v_x(\xi')}{1 + \xi'^2} d\xi' + \chi(\xi). \end{aligned} \quad (2.26)$$

A typical solution is shown in Fig. 2.2. The velocity amplitudes is represented by the derivatives of the magnetic fields

$$v_z = -\frac{1}{\tilde{K}} d_\xi b_z, \quad v_x = -\frac{1}{\tilde{K}} d_\xi b_x, \quad v_y = \frac{K_\perp}{K_y} \xi v_x - \frac{K_x}{K_y} v_z, \quad (2.27)$$

this is actually a consequence of the Alfvén theorem that magnetic field is frozen in a fluid with negligible Ohmic resistivity.

For incompressible fluid the difference between AW and SMW is only in polarization, but effect of amplification of SMW exists even if compressibility is taken into account.

We consider propagation of plane waves in a homogeneous medium. The variables $\mathbf{v}(\xi)$ and $\mathbf{b}(\xi)$ are time dependent but for zero dissipation at infinite time we have well defined asymptotics of the amplitudes. At those infinite times we have fixed frequency which corresponds to the dispersion of SMW. In such a way the analytical solution describes the amplification of SMW from $-\infty$ to $+\infty$.

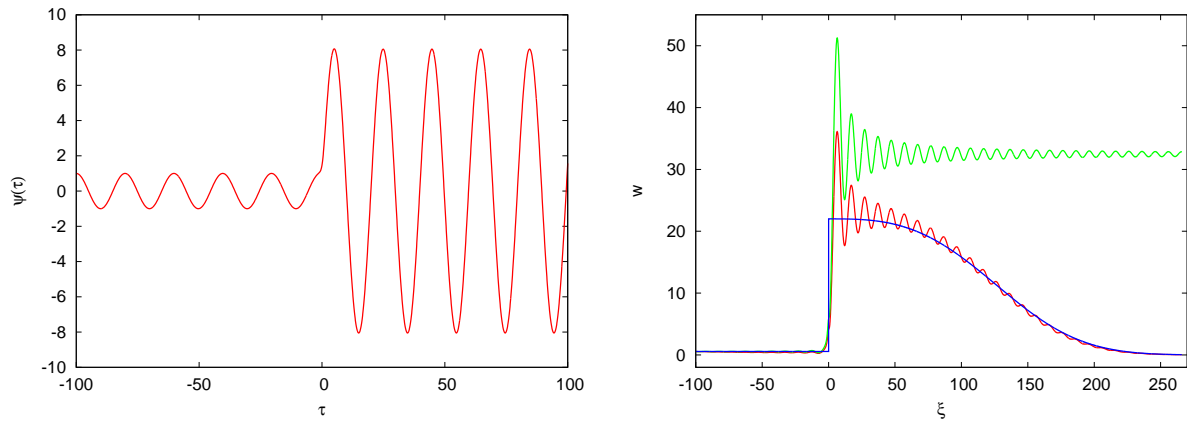


Figure 2.2: *Left*: Solution of effective Schroedinger equation for $\psi(-100) = 1$, $d_\tau \psi(-100) = 0$, and $K_\perp = \sqrt{0.1}$. When $\tau = 0$ is observed almost saltatory increase of the amplitude of oscillation which is related to the amplification of SMW. The analytical solution in framework of Heun functions reproduces the amplification discovered by numerical analysis [Chagelishvili et al.(1993)]. *Right*: Wave energy as a function of time with damping pulsations around the average value. One can see that viscosity leads to attenuation of the waves and further heating of plasma. Enveloping curve corresponds to $\exp \left[- \int \nu k^2(t) dt \right]$. The continuous line demonstrates that attenuation of the waves for the case of small dissipation is well-described by WKB approximation.

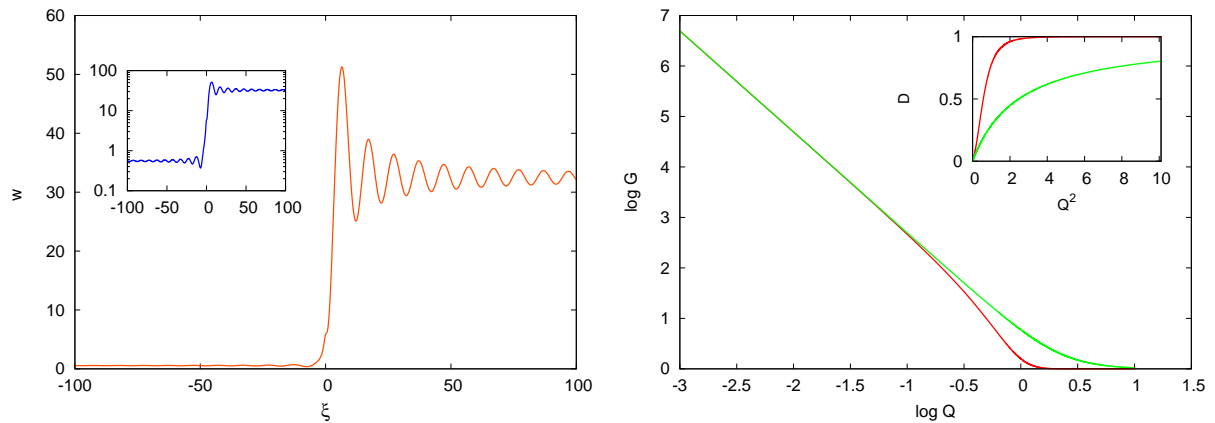


Figure 2.3: *Left*: wave energy as a function of time. On inset figure energy of wave is shown in logarithmic scale. *Right*: logarithm of the ratio of amplification of waves as a function of the logarithm of the wave vector. Simple asymptotic formula Eq. (2.81) (right curve) accurately describes long-wavelengths asymptote of the analytical solution. The left hand side analytically reproduces the energy amplification observed for first time by the numerical calculations. The right hand side is new and reproduces the long-wavelength asymptotic for the amplification coefficient.

A typical time dependence of the wave energy density $w(\xi)$ is given in Fig. 2.3 (left). For the case of in-plane static magnetic field $B_{0,z} = 0$ the amplification of the SMW is directly determined by the amplification of the amplitude of the ψ -function for which we have the asymptotic derived by the δ -function method

$$\mathcal{G}(|K_\perp| \ll 1) \approx \frac{\pi^2}{2K_\perp^2} + 1 = \frac{1}{2} \frac{\pi^2 A^2}{V_A^2(k_\varphi^2 + k_z^2)} + 1 \gg 1, \quad (2.28)$$

and the details of derivation are given in [?].

On the right of the same figure asymptotic formula for amplification at large wavelengths and the exact solution are presented. Formally the use of this asymptotic is reduced to the application of the method of δ -like potential approximation in quantum mechanics, in our case to substitution

$$\frac{1}{(1 + \xi^2)^2} \rightarrow \pi \delta(\xi) \quad (2.29)$$

in the effective Schrödinger equation Eq. (2.21). The asymptotic accuracy of the δ -potential method at long-wave limit reveals that the phenomenon of amplification of SMW cannot be understood in the framework of WKB method.

The presence of small viscosity leads to damping of waves determined by the total viscosity $\nu_{\text{tot}} = \nu_k + \nu_m$, and we have damping factor $\exp \left[- \int \nu_{\text{tot}} k^2(t) dt \right]$ which leads to the time dependence of the amplitude

$$\psi \approx D_f \theta(\tau) \cos(K_y \tau + \phi_f) \exp \left(-\nu'_{\text{tot}} K_y^2 \tau^3 / 3 \right), \quad \nu'_{\text{tot}} \ll 1, \quad |\tau| \gg 1. \quad (2.30)$$

The influence of viscosity on the waves is illustrated in Fig. 2.2-Right, where two solutions with zero, nonzero viscosity, and enveloping simple exponential function are depicted. In short we revealed that wave amplification can be observed even without any rotation and very large amplification factors can be not related to MRI. Once amplified the Alfvén waves decay as independent free waves; one can say that they are asymptotically free.

The derived asymptotic formula for the gain Eq. (2.81) gives infinite wave amplification for $K_y = 0$. As we will see in the next section this axisymmetric case for the accretion disks give the well known MRI for nonzero rotation and axial magnetic field.

2.6 Amplification of the MHD Waves

The odd and even solutions have asymptotics at $\xi \rightarrow \infty$

$$\psi_g \approx D_g \cos(K_\perp \xi + \delta_g), \quad \psi_u \approx D_u \cos(K_\perp \xi + \delta_u), \quad (2.31)$$

where the asymptotic phase shifts $\delta_g(K_\perp^2)$, $\delta_u(K_\perp^2)$ depend on the effective energy. For a sufficiently large wave-vectors we have the asymptotic

$$\delta_g(K_\perp^2 \gg 1) \approx 0, \quad \delta_u(K_\perp^2 \gg 1) \approx -\frac{\pi}{2}. \quad (2.32)$$

As we will see later, the averaged amplification coefficient of the energy of MHD waves $\mathcal{A}(\delta_g, \delta_u)$ depends only on the asymptotic phases of the solutions but not on the amplitudes D_g and D_u .

2.6.1 Auxiliary Quantum Mechanical Problem

Temporarily introducing imaginary exponents $\exp(i\mathbf{k} \cdot \mathbf{r})$ instead of $\sin(\mathbf{k} \cdot \mathbf{r})$ and $\cos(\mathbf{k} \cdot \mathbf{r})$ gives significant simplification of the analytical calculations related to the physics of the waves. In this subsection we will consider ψ in as a complex function in order to make easier any further analysis of the MHD amplification coefficient. In order to analyze the effective MHD equation we will solve the quantum-mechanical counterpart of our MHD problem, that is a tunneling through a barrier $2mU/\hbar^2 = 1/(1 + \xi^2)^2$, supposing that ψ is a complex function. Consider an incident wave with a unit amplitude, a reflected wave with amplitude R and a transmitted wave with amplitude T

$$\psi(\xi \rightarrow -\infty) \approx \exp(iK_\perp \xi) + R \exp(-iK_\perp \xi), \quad (2.33)$$

$$\psi(\xi \rightarrow +\infty) \approx T \exp(+iK_\perp \xi). \quad (2.34)$$

Using the asymptotics of the eigen-functions

$$\psi_g \approx \begin{cases} D_g \cos(K_\perp \xi - \delta_g) & \text{for } \xi \rightarrow -\infty, \\ D_g \cos(K_\perp \xi + \delta_g) & \text{for } \xi \rightarrow +\infty, \end{cases} \quad (2.35)$$

and

$$\psi_u \approx \begin{cases} -D_u \cos(K_\perp \xi - \delta_u) & \text{for } \xi \rightarrow -\infty, \\ D_u \cos(K_\perp \xi + \delta_u) & \text{for } \xi \rightarrow +\infty \end{cases} \quad (2.36)$$

as well as the general condition

$$\psi(\xi) = C_g^{(q)} \psi_g(\xi) + C_u^{(q)} \psi_u(\xi), \quad (2.37)$$

we compare the coefficients in front of $\exp(iK_\perp \xi)$ and $\exp(-iK_\perp \xi)$ for $\xi \rightarrow -\infty$ and $\xi \rightarrow \infty$. The solution to a simple matrix problem yields

$$C_g^{(q)} D_g = \exp(i\delta_g), \quad C_u^{(q)} D_u = -\exp(i\delta_u) \quad (2.38)$$

Then for the tunneling amplitude we get

$$T = |T| e^{i(\delta_u + \delta_g - \pi/2)} \quad (2.39)$$

and finally for the tunneling coefficient we obtain

$$\mathcal{D} = |T|^2 = s_{ug}^2, \quad s_{ug} \equiv \sin(\delta_u - \delta_g). \quad (2.40)$$

The convenience of the tunneling coefficient is that it varies in the range $0 \leq \mathcal{D} \leq 1$. In the next two subsections we will present the SMW amplification coefficient as a function of the tunneling

coefficient $\mathcal{K} = 2/\mathcal{D} - 1$ on the analogy of Heisenberg's ideas in quantum mechanics where the statistical properties of the scattering problem depend only on the phases of the S -matrix.

The phases $\delta_g, \delta_u \in (-\pi, \pi)$ and amplitudes D_g and D_u can be determined continuing the exact wave functions with WKB asymptotics Eq. (2.31). Thus we obtain

$$\begin{aligned} \psi &= D \cos(K_\perp \xi + \delta), \quad \tilde{\delta} = -K_\perp \xi - \arctan \frac{d_\xi \psi(\xi)}{K_\perp \psi(\xi)}, \\ \delta &= \tilde{\delta} - 2\pi \times \text{int} \left(\frac{\tilde{\delta} + \pi}{2\pi} \right) \in (-\pi, \pi), \quad D = \frac{\psi(\xi)}{\cos(K_\perp \xi + \delta)}, \end{aligned} \quad (2.41)$$

at some sufficiently large $\xi \gg 1 + 2\pi/K_\perp$. Here $\text{int}(\dots)$ stands for the integer part of a real number. When programming we have to use the two-argument arctan function

$$\arctan(y, x) = \arctan(y/x) + \frac{\pi}{2} \theta(-x) \text{sgn}(y) \in (-\pi, \pi). \quad (2.42)$$

The accuracy of this continuation is controlled by the Wronskian from the asymptotic wave functions

$$W(\psi_g, \psi_u)(\xi) = Q D_g D_u \sin(\delta_g - \delta_u) = 1. \quad (2.43)$$

2.6.2 MHD and Real ψ

Imagine that in an ideal plasma we have at $t \rightarrow -\infty$ some plane MHD wave – our task is to calculate how many times the energy density increases at $t \rightarrow \infty$, and to average this amplification over all the initial phases of that wave. As there is no amplification for the b_z component according to Eq. (2.25) we will concentrate our attention on the b_x component. The amplification comes from the negative “friction” term $\propto \xi/(1 + \xi^2)$. The influence of this friction is transmitted to the effective potential barrier $\propto 1/(1 + \xi^2)$. For $K_\perp^2 < 1$ we have an analog of the quantum mechanical tunneling.

In the current MHD problem ψ is a real variable with asymptotics

$$\psi \approx \begin{cases} \cos(K_\perp \xi - \phi_i) & \text{for } \xi \rightarrow -\infty, \\ D_f \cos(K_\perp \xi + \phi_f) & \text{for } \xi \rightarrow +\infty, \end{cases} \quad (2.44)$$

i.e., we have an incident wave with a unit amplitude and an initial phase ϕ_i . D_f is the amplitude and ϕ_f is the phase of the amplified wave.

Again we present the ψ function as linear combination of even and odd solutions

$$\psi(\xi) = C_g^{(c)} \psi_g(\xi) + C_u^{(c)} \psi_u(\xi). \quad (2.45)$$

Here we substitute the asymptotic formulas Eq. (2.35) and Eq. (2.8), and the comparison of the coefficient with Eq. (2.8) at $\xi \rightarrow -\infty$ gives

$$C_g^{(c)} D_g = \frac{S_{iu}}{S_{gu}}, \quad C_u^{(c)} D_u = \frac{S_{ig}}{S_{gu}}, \quad (2.46)$$

where

$$s_{iu} \equiv \sin(\phi_i - \delta_u), \quad s_{ig} \equiv \sin(\phi_i - \delta_g). \quad (2.47)$$

The comparison of the coefficients at $\xi \rightarrow \infty$ gives for the phase and the amplification of the signal

$$\phi_f = F(\phi_i) \equiv \arctan \frac{s_{ig}s_u + s_{iu}s_g}{s_{ig}c_u + s_{iu}c_g}, \quad (2.48)$$

$$\mathcal{A}(\phi_i) \equiv D_f^2 = \frac{\mathcal{N}}{\mathcal{D}}, \quad (2.49)$$

where

$$\mathcal{N} = (s_{ig}s_u + s_{iu}s_g)^2 + (s_{ig}c_u + s_{iu}c_g)^2, \quad (2.50)$$

$$s_g = \sin(\delta_g), \quad s_u = \sin(\delta_u), \quad (2.51)$$

$$c_g = \cos(\delta_g), \quad c_u = \cos(\delta_u). \quad (2.52)$$

The reversibility of the dissipation-free motion leads us to $\phi_i = F(\phi_f)$, i.e., function F coincides with its inverse function $F(F(\phi)) = \phi$. As time reverses the wave amplification is converted to attenuation (damping in some sense) $\mathcal{A}(\phi_i)\mathcal{A}(F(\phi_i)) = 1$.

In unabridged mathematical notations we have the function

$$F(\varphi) \equiv \arctan \frac{\sin(\varphi - \alpha) \sin \beta + \sin(\varphi - \beta) \sin \alpha}{\sin(\varphi - \alpha) \cos \beta + \sin(\varphi - \beta) \cos \alpha}, \quad (2.53)$$

defined in the interval $\varphi \in (-\pi/2, \pi/2)$. For arbitrary values of the parameters α and β

$$F[F(\phi)] = \phi, \quad (2.54)$$

i.e., this function F coincides with its inverse function F^{-1} . The nonlinear function F has only 2 immovable points

$$F(\alpha) = \alpha, \quad F(\beta) = \beta. \quad (2.55)$$

Defining also

$$\mathcal{A}(\varphi) \equiv \{[\sin(\varphi - \alpha) \sin \beta + \sin(\varphi - \beta) \sin \alpha]^2 + [\sin(\varphi - \alpha) \cos \beta + \sin(\varphi - \beta) \cos \alpha]^2\} / \sin^2(\alpha - \beta),$$

we have another curious relation

$$\mathcal{A}[F(\varphi)] \mathcal{A}(\varphi) = 1. \quad (2.56)$$

The so derived amplification coefficient $\mathcal{A}(\phi_i; \delta_g, \delta_u)$ depends on the initial phase. In the next subsection we will consider the statistical problem of phase averaging with respect to the initial phase ϕ_i .

2.6.3 Phase Averaged Amplification

For waves generated by turbulence the initial phase is unknown and one can suppose a uniform phase distribution. That is why for solving the statistical problem of energy amplification we need to calculate average values with respect to the initial phase ϕ_i . That idea is coming from the well-known random phase approximation (RPA) in plasma physics. The phase averaging already introduces an element of irreversibility because we already suppose that waves are created with random phases. This is the MHD analog of the molecular chaos from the theorem for entropy increase in the framework of the kinetic theory if the probability distributions are introduced in the initial conditions of the mechanical problem. In the case of accretion flows we also suppose that turbulence is a chaotic phenomenon and we have to apply the RPA for investigating the statistical properties.

The calculation of the integral

$$\int_0^\pi \frac{d\phi_i}{\pi} \mathcal{N}(\phi_i) = 2 - s_{\text{ug}}^2 = 2 - \mathcal{D} \quad (2.57)$$

gives for the initial phase averaged gain

$$\mathcal{G} \equiv \int_{-\pi/2}^{\pi/2} \mathcal{A}(\phi_i) \frac{d\phi_i}{\pi} = \frac{2}{\sin^2(\delta_u - \delta_g)} = \frac{2}{\mathcal{D}} - 1. \quad (2.58)$$

In such a way the SMW amplification coefficient \mathcal{G} is presented by the tunneling coefficient \mathcal{D} of the corresponding quantum problem. Both coefficients are expressed by the asymptotic phases δ_g, δ_u in analogy with partial waves phase analysis of the quantum mechanical scattering problem in atomic and nuclear physics. The axial symmetry of this result $\mathcal{G}(\sqrt{K_y^2 + K_z^2})$ significantly simplifies the further statistical analysis.

Let us analyze the physical meaning of the gain coefficient \mathcal{G} . As

$$K_\perp \xi = K_y \tau = V_A k_y t = \omega_A \text{sgn}(k_y) t, \quad \omega_A(\mathbf{k}) = |\mathbf{V}_A \cdot \mathbf{k}| = V_A |k_y| \geq 0, \quad (2.59)$$

$$\mathbf{v}_{\text{gr}} \equiv \frac{\partial \omega_A}{\partial \mathbf{k}} = \mathbf{V}_A \text{sgn}(\mathbf{V}_A \cdot \mathbf{k}) = V_A \text{sgn}(k_y) \mathbf{e}_y, \quad (2.60)$$

the asymptotics Eq. (2.8) let us conclude that for $|t| \rightarrow \infty$ we have only magnetosonic waves with dispersion coinciding with that of Alfvén waves. [Alfvén (1942)] In the spirit of M. T. Weiss quantum interpretation of the classical Manley–Rowe theorem [Landau and Lifshitz (1983), Zhelyazkov (2000)] one can present the wave energy $\hbar \omega_A N$ by a number of quanta, the number of *alfvenons*: “The alfvénons introduced in this Letter [Stasiewicz (2006)] appear to be effective and spectacular converters of electromagnetic energy flux into kinetic energy of particles.” We use this notion in a slightly different sense, our former terminology was *alfvons* [Mishonov et al. (2007)] in our case. Following this interpretation, the energy gain \mathcal{G} describes the increasing number of quanta

$$\mathcal{G}(\sqrt{k_y^2 + k_z^2}) = \frac{\hbar \omega_A(|k_y|) \overline{N}_{\text{alfvenons}}(t \rightarrow +\infty)}{\hbar \omega_A(|k_y|) \overline{N}_{\text{alfvenons}}(t \rightarrow -\infty)}, \quad (2.61)$$

as in a laser system. In this terminology the mechanism of heating of quasars can be phrased, namely due to lasing of alfvénons in shear flows of magnetized plasma. Laser or rather maser [Trakhtengerts (2008)] effects are typical phenomena in space plasmas. The hydrodynamic over-reflection instability [Fridman and Bisikalo (2008)] and burst-like increase of the wave amplitude [Rogava et al. (2003)] are phenomena of similar kind. More precisely \mathcal{G} is the gain for the x - y -polarized SMWs, the energy of mode conversion in z -polarized AWs will be analyzed elsewhere. The notion amplification is correct for standing waves but $2 \cos(\mathbf{k} \cdot \mathbf{r}) = e^{i\mathbf{k} \cdot \mathbf{r}} + e^{-i\mathbf{k} \cdot \mathbf{r}}$ and amplification is simultaneous for opposite wave-vectors. That is why some theorists prefer to use overreflection in spite that there is no rigid object reflecting the waves.

2.6.4 Analytical Approximations for Amplification

For scattering problems by a localized potential at small wave-vectors $K_\perp \ll 1$, when the wavelength is much larger than the typical size of the nonzero potential, we can apply the delta-function approximation

$$\frac{2m}{\hbar^2} U(\xi) \equiv \frac{1}{(1 + \xi^2)^2} \rightarrow 2Q_0 \delta(\xi). \quad (2.62)$$

In this well-known quantum mechanical problem [Greiner (2001)] the transmission coefficient is

$$D \approx \frac{K_\perp^2}{K_\perp^2 + Q_0^2}, \quad \delta_u = -\frac{\pi}{2}, \quad \delta_g = -\frac{\pi}{2} - \arctan \frac{K_\perp}{Q_0}, \quad (2.63)$$

$$\psi_g = \frac{\cos(|K_\perp \xi| + \delta_g)}{\cos(\delta_g)}, \quad \psi_u = \frac{\sin K_\perp \xi}{K_\perp}. \quad (2.64)$$

According to the tradition of the method of potential of zero radius, the coefficient $Q_0 \approx \frac{1}{2}\pi$ is determined by the behavior of the phases at small wave-vectors. Only qualitatively this parameter corresponds to the area of the potential

$$2Q_0 = \frac{\pi}{2} Z_{\text{ren}} = \pi, \quad \frac{\pi}{2} = \int_{-\infty}^{\infty} \frac{1}{(1 + \xi^2)^2} d\xi, \quad (2.65)$$

but the renormalizing coefficient $Z_{\text{ren}} = 2$ which we have to introduce differs from unity. According to the general relation Eq. (2.58) for the amplification we have

$$\mathcal{G} - 1 = 2 \left(\frac{1}{\mathcal{D}} - 1 \right) \approx \frac{2Q_0^2}{K_\perp^2}, \quad 2Q_0^2 = \frac{\pi^2}{2} \approx 4.934 \approx 5. \quad (2.66)$$

The scattering phases δ_g and δ_u can be obtained by a fit of the asymptotic wave functions Eq. (2.35) and Eq. (2.8) with the analytical solutions. In such a way we can calculate the wave-vector dependence of the transmission coefficient as it is depicted in.

Let us derive the analytical formula for Q_0 : The solutions

$$\psi_g(\xi) = \sqrt{1 + \xi^2}, \quad (2.67)$$

$$\psi_u(\xi) = \sqrt{1 + \xi^2} \arctan(\xi) \quad (2.68)$$

of effective Schrödinger's equation in long wavelength limit $K_\perp \rightarrow 0$

$$d_\xi^2 \psi - \frac{1}{(1 + \xi^2)^2} \psi = 0 \quad (2.69)$$

with asymptotics at $\xi \rightarrow \infty$

$$\psi_g(\xi \gg 1) \approx \xi, \quad (2.70)$$

$$\psi_u(\xi \gg 1) \approx \frac{\pi}{2} \xi \quad (2.71)$$

have to be compared with the approximative solutions Eq. (2.35) and Eq. (2.8) for $\xi \gg 1$, when for $K_\perp \rightarrow 0$ we have $|\delta_g| \approx |\delta_u| \approx \frac{\pi}{2}$ and sinusoids are almost linear

$$\psi_g(1 \ll \xi \ll \frac{1}{K_\perp}) \approx D_g \sin(K_\perp \xi) \approx D_g K_\perp \xi, \quad (2.72)$$

$$\psi_u(1 \ll \xi \ll \frac{1}{K_\perp}) \approx D_u \sin(K_\perp \xi) \approx D_u K_\perp \xi. \quad (2.73)$$

The comparison of the first derivatives in this region

$$d_\xi \psi_g(\xi \gg 1) \approx 1 \approx D_g K_\perp, \quad (2.74)$$

$$d_\xi \psi_u(\xi \gg 1) \approx \frac{\pi}{2} \approx D_u K_\perp \quad (2.75)$$

gives

$$D_g \approx \frac{1}{K_\perp}, \quad D_u \approx \frac{\pi}{2K_\perp}. \quad (2.76)$$

Then formula for Wronskian Eq. (2.43) determines the phase difference

$$\sin(\delta_g - \delta_u) \approx \frac{2}{\pi} K_\perp \quad (2.77)$$

and transmission coefficient

$$\mathcal{D} \approx \sin^2(\delta_g - \delta_u) = \left(\frac{2K_\perp}{\pi} \right)^2. \quad (2.78)$$

The Eq. (2.58) then gives for the phase averaged amplification

$$G = \frac{2}{D} - 1 \approx \frac{\pi^2}{2Q^2} + \text{const} \quad (2.79)$$

and the comparison with δ -potential approach Eq. (2.66) gives the analytical result for the strength of the δ -potential $Q_0 = \pi/2$ given in Eq. (2.66).

Our problem formally coincides with the quantum problem [Landau and Lifshitz (1932)] of transmission coefficient $\mathcal{D} \propto E$ at low energies

$$U(x) = \frac{U_0}{[1 + (\alpha x)^2]^2}, \quad U_0 = \frac{\hbar^2 \alpha^2}{2m}, \quad E = U_0 K_\perp^2, \quad D \approx \left(\frac{2K_\perp}{\pi} \right)^2 = \frac{4}{\pi^2} \frac{E}{U_0} \ll 1. \quad (2.80)$$

The zero-radius potential Padé approximant Eq. (2.63) has an acceptable accuracy for small wave-vectors. Having the quantum mechanical transmission coefficient, we can calculate the energy gain coefficient $\mathcal{G} - 1$.

$$\mathcal{G} - 1 \sim \frac{1}{K_{\perp}^2} = \frac{A^2}{V_A^2 q^2}, \quad q^2 = k_{\perp}^2 \equiv k_y^2 + k_z^2 = \text{const} \quad (2.81)$$

Here the subscript \perp means that the amplification depends on the projection of the wave-vector perpendicular to the shear velocity and magnetic field.

The delta function approximation has a visual interpretation in classical mechanics as well, supposing that ψ is the displacement of an oscillator and ξ is the time. The approximative equation

$$d_{\xi}^2 \psi = -K_{\perp}^2 \psi(\xi) + 2Q_0 \delta(\xi) \psi(\xi) \quad (2.82)$$

means that at a time moment $\xi = 0$ the oscillator is subjected to a forcing impulse with a magnitude

$$d_{\xi} \psi(+0) - d_{\xi} \psi(-0) = 2Q_0 \psi(0), \quad \text{or} \quad v_x(+0) - v_x(-0) = \frac{\pi b_x(0)}{\sqrt{K_y^2 + K_z^2}}, \quad \text{when} \quad K_x(0) = 0.$$

If for $K_{\perp} \ll Q_0$ we have initial oscillations with amplitude A_i

$$\psi = A_i \cos K_{\perp} \xi \quad \text{for} \quad \xi < 0, \quad (2.83)$$

after the push in $\xi = 0$ we have oscillations with much increased amplitude

$$\psi = A_f \sin K_{\perp} \xi, \quad \text{for} \quad \xi > 0, \quad A_f \approx \frac{2Q_0}{K_{\perp}} A_i \gg A_i. \quad (2.84)$$

The strong push with an appropriate phase “amplifies” the oscillations; this burst-like increase of the wave amplitude was observed in numerical investigations of linearized two-dimensional MHD equations. [Chagelishvili et al.(1993), Rogava et al. (2003)] This phenomenon is akin to the extremely strong hydrodynamic instabilities due to a velocity jump; its prediction and discovery both in theory and experiments are described in Ref. [Fridman et al. (2008a)]. In such a way the Alfvén’s idea of the importance of MHD waves in the transfer of momentum is reduced to the very simple mathematics for the jump of the velocity of SMWs Eq. (2.6.4). It is instructive to rewrite this force in the r-space.

2.7 MRI and deviation equation as test example of the general equation

2.7.1 Secular equations and Lyapunov analysis

The axisymmetrical case $K_y = 0$ has measure zero but it is very instructive to reveal Velikhov [Velikhov (1959)] MRI which with some delay become one of the most popular notion

in astrophysics. For $K_y = 0$ we have $\mathbf{U}_{\text{shear}} = 0$, $\mathbf{K} = \text{const}$, and $D_\tau^{\text{shear}} = d_\tau$. In this case the linearized MHD equations Eq. (2.13) and Eq. (2.14) are a linear system with constant coefficients having exponential time dependence $\propto \exp(\lambda\tau)$.

Supposing that for small $|K_y| \ll 1$ we have small shear velocity we have slowly changing by time wave-vector

$$\mathbf{K}(\tau) = \mathbf{K}_0 + (\tau - \tau_0)\mathbf{U}_{\text{shear}}. \quad (2.85)$$

The substitution of this $\mathbf{K}(\tau)$ in the system above gives the WKB approximation of slowly changing coefficients. For ideal fluid $\nu'_k = 0 = \nu'_k$ the corresponding secular equation simplifies as

$$\begin{aligned} \lambda^4 - 2n_y n_x \lambda^3 + \{[(4 - 8n_y^2)n_x^2 + 4 - 4n_y^2]\omega_c^2 + [(2 - 8n_y^2)n_x^2 - 4n_y^2 + 2]\omega_c + 2K_\alpha^2\}\lambda^2 \\ - 2K_\alpha^2 n_y n_x \lambda + 2K_\alpha^2(n_x^2 + 1)\omega_c + K_\alpha^4 = 0. \end{aligned} \quad (2.86)$$

Let $\lambda_{3D}(\mathbf{K})$ is the solution with maximal real part. It embraces all cases of MRI analyzed in the famous works by Balbus and Hawley [Balbus and Hawley Part I (1991), Balbus and Hawley Part II (1991), Balbus and Hawley Part III (1992), Balbus and Hawley Part IV (1992), Balbus and Hawley (1992), Balbus and Hawley (1998)].

For the 2-dimensional axi-symmetric case we have growth rate

$$\lambda_{2D}(K_x, K_z) = \lambda_{3D}(K_x, K_y = 0, K_z), \quad (2.87)$$

which obeys already exact the equation

$$\lambda^4 + 2[K_\alpha^2 + (1 + 2\omega_c)(n_x^2 + 1)\omega_c]\lambda^2 + 2K_\alpha^2(n_x^2 + 1)\omega_c + K_\alpha^4 = 0. \quad (2.88)$$

The most restricted is the 1-dimensional case $\lambda_{1D}(K_z) = \lambda_{2D}(K_x = 0, K_z)$ with wave-vector parallel to the rotation axis $\mathbf{K} = K\mathbf{e}_z$ when $K_\alpha = K_z \cos \theta$ when we have the most cited bi-quadratic equation in the astrophysics

$$\lambda^4 + 2[K_\alpha^2 + (1 + 2\omega_c)\omega_c]\lambda^2 + (K_\alpha^2 + 2\omega_c)K_\alpha^2 = 0, \quad (2.89)$$

which describes the Velikhov MRI growth rate.

In order to compare the results we have to use the traditional for the physics of disks dimensional variables. Having in orbital velocity of the fluid $V_\varphi(r) = r\Omega(r)$ the shear describes the component of the viscous stress tensor

$$\sigma_{r\varphi} = \eta \left(\frac{1}{r} \partial_\varphi V_r + \partial_r V_\varphi - \frac{V_\varphi}{r} \right) = \eta A. \quad (2.90)$$

In such a way we can define the shear rate and dimensional angular velocity by the radial dependence of the angular velocity

$$A(r) = r d_r \Omega \approx \text{const.}, \quad \omega_c = \frac{\Omega}{A} = \frac{d \ln r}{d \ln \Omega}. \quad (2.91)$$

For Keplerian disks when $\Omega \propto r^{-3/2}$ we have $\omega_c = -\frac{2}{3}$, while at solar tachocline we have regions with $|\omega_c| \ll 1$, whose linearized MHD is well described by the analytical solution of $\omega_c = 0$ considered in the former section 2.5.

Further we define the usual formulas for: the Alfvén frequency

$$\omega_A = |\mathbf{V}_A \cdot \mathbf{k}| = V_A k_z \cos \theta, \quad (2.92)$$

where \mathbf{k} is the usual wave-vector, the epicyclic frequency [Balbus and Hawley Part I (1991), Balbus and Hawley (1998)]

$$\kappa^2 \equiv \frac{1}{r^3} \frac{d(r^4 \Omega^2)}{dr} = \frac{2\Omega}{r} \frac{d}{dr} (r^2 \Omega) = 2\Omega \cdot \text{rot} [r\Omega(r)\mathbf{e}_\varphi] = 2A\Omega + 4\Omega^2, \quad (2.93)$$

and the usual frequency $\tilde{\omega} = iA\lambda$. In these variables the dispersion relation for MHD modes Eq. (2.89) takes the well known form [Balbus and Hawley Part I (1991), Balbus and Hawley (1998)]

$$\tilde{\omega}^4 - \tilde{\omega}^2 (\kappa^2 + 2\omega_A^2) + \omega_A^2 \left(\omega_A^2 + \frac{d\Omega^2}{d \ln r} \right) = 0, \quad \frac{d\Omega^2}{d \ln r} = 2A\Omega, \quad (2.94)$$

with the solution

$$\tilde{\omega}_\pm^2(k_z) = \frac{1}{2} \left[\kappa^2 + 2\omega_A^2 \pm \sqrt{\kappa^4 + 16\omega_A^2 \Omega^2} \right], \quad (2.95)$$

which describes the growth rate of MRI when $\tilde{\omega}_-^2 < 0$. The rederivation of the growth rate of MRI is an important test for the validity of our new basic equations Eq. (2.6) and Eq. (2.7). For a short review we mention: the momentum dependence of $\lambda_{2D}^2(K_x, K_z)$ was depicted in Fig. 1a of [Balbus and Hawley Part I (1991)], the dispersion relation was given in Eq. (2.9) and Eq. (2.17). The momentum dependence of $\lambda_{2D}(K_x = \text{const}, K_z)$ was given in the Fig. 8 in [Balbus and Hawley Part II (1991)] together with numerical analysis performed in coordinate space. In the same work authors came to the conclusion that compressibility has no significant effect on the instability. Graphical dependence of $\lambda_{2D}(K_x = \text{const}, K_z)$ was presented also in Fig. 1 by [Balbus and Hawley Part III (1992)]; and on Fig. 7 was presented the section in the other direction $\lambda_{2D}(K_x, K_z = \text{const})$. The nonaxisymmetric dynamic equation Eq. (2.25) corresponding to our Eq. (2.86) was given in [Balbus and Hawley Part IV (1992)]. In this article Balbus and Hawley pointed out the important connection, at the linear level, between the MRI and the giant amplification of SMW in the case of Coriolis force. Their WKB formula at the bottom of page 616 of Ref. [Balbus and Hawley Part IV (1992)] in our notations for the energy gain reads as

$$G \approx \exp \left(2 \int_{-\infty}^{\infty} d\tau \lambda_{3D}(\mathbf{K}(\tau)) \right) = \exp \left(2 \int_{-\infty}^{\infty} dQ_x \lambda_{3D}(\mathbf{Q})/|Q_y| \right) \stackrel{\text{def}}{=} e^{\mu(Q_y, Q_z)/|Q_y|}, \quad (2.96)$$

the last equation is the definition for $\mu(Q_y, Q_z)$.

Strictly speaking for non axisymmetric case $K_y \neq 0$ MRI does not exists because we have a linear system with *time dependent coefficients*. The $\exp(\lambda\tau)$ time dependence is a property

of differential equations with constant coefficients. However the WKB approximation is good working for Keplerian values of the angular velocity and “amplification factors can be tens of orders of magnitude”– page 620 of [Balbus and Hawley Part IV (1992)]. In order to emphasize that SMW can grow by very large factors, Balbus and Hawley put quotation marks at page 618 and use the notion “instabilities,” even though the modes “are technically transient amplification.” Really for very small $|Q_y| \ll 1$ we have $\lambda_{3D}(\mathbf{Q}) \approx \lambda_{2D}(Q_x, Q_z)$. In this approximation for the energy gain we obtain

$$G(Q_y, Q_z) \approx \exp \left(2 \int_{-\infty}^{\infty} dQ_x \lambda_{2D}(Q_x, Q_z) / |Q_y| \right) = e^{\mu(0, Q_z) / |Q_y|}. \quad (2.97)$$

The dependence $\exp(2\mu/|Q_y|)$ presents how the amplification coefficient of SMW can become giant in the linear regime, for $Q_y \rightarrow 0$. One of the main results of our work is the above relation between the amplification of magnetosonic waves and growth rate of the MRI. This relation is actually WKB approximation and in this sense we can speak about the WKB SMW amplification. Divergence of the amplification coefficient G at $Q_y \rightarrow 0$ means that giant amplification for $|Q_y| \ll 1$ is the precursor of MRI at $Q_y = 0$. The path of the wave-vector trough the unstable regions is traced in Fig. 1 of [Balbus and Hawley Part IV (1992)] and Fig. 17 of [Balbus and Hawley (1998)], where the most of the result from the original ApJ papers are reviewed. If we trace the wave-vector trajectory $\mathbf{K}(\tau)$ substantial growth is occur for a finite time spent in the unstable region. The physics is analogous to the wave amplification by a lasing medium. Picturesquely speaking the enhanced transport in accretion disks is created by the lasing of alfvénons. The giant amplification closer to $Q_y = 0$ plane is the mechanism which trigger the self consistent turbulence and anomalous transport in accretion disks. In order to emphasize the importance of the exponential growth Balbus and Hawley called it “instability” because it is believed that in many physical situations the nonlinear effects become important very fast and this smears the difference between the amplification and instability. That is why one can also say that MRI is the “heart” of turbulence [Balbus and Hawley (1998)].

However divergence of the amplification is not a property of the rotation. Without rotation the infinite amplification of SMW for $K_{\perp} = 0$ was discovered [Chagelishvili et al.(1993)] even before MRI to be exhumed for the astrophysics by [Balbus and Hawley Part I (1991)]. This amplification is adequately described by a δ -function like sudden change of the rigidity Eq. (2.29) of an effective oscillator is called “swing” amplification by [Fan and Lou (1997)].

We demonstrated that general equations given in the present work describe both the “swing” SMW amplification and WKB SMW amplification called sometimes MRI. As further test of the new approach we will consider in the next sub-section how the [Balbus and Hawley (1998)] mechanical model for MRI can be re-derived as a consequence of our general MHD equations.

2.7.2 Deviation equation

In order to illustrate derivation of the deviation equation we will consider only the simplest possible case of axial wave-vector in ideal fluid. The linearized MHD equations Eq. (2.13) and

Eq. (2.14) in this case take the form

$$d_\tau \mathbf{v} = -v_x \mathbf{e}_y + (\boldsymbol{\alpha} \cdot \mathbf{K}) \mathbf{b} + 2\omega_c (v_y \mathbf{e}_x - v_x \mathbf{e}_y), \quad (2.98)$$

$$d_\tau \mathbf{b} = b_x \mathbf{e}_y - (\boldsymbol{\alpha} \cdot \mathbf{K}) \mathbf{v}. \quad (2.99)$$

In order to eliminate the magnetic field we express \mathbf{b} from Eq. (2.98) and substitute in Eq. (2.99). In the obtained vector equation we present the velocity by the displacement $\mathbf{v} = d_t \boldsymbol{\xi} = \dot{\boldsymbol{\xi}}$. One additional time integration gives

$$\begin{aligned} \ddot{\xi}_r - 2\Omega \dot{\xi}_\varphi &= -\omega_A^2 \xi_r, \\ \ddot{\xi}_\varphi + 2\Omega \dot{\xi}_r &= -(\omega_A^2 + 2A\Omega) \xi_\varphi. \end{aligned} \quad (2.100)$$

The mechanical model [Balbus and Hawley (1998), Balbus (2003)] for this system is an anisotropic oscillator in rotating coordinate system.

The variables $\xi_r(t)$ and $\xi_\varphi(t)$ are exactly the amplitudes of the wave induced deviation ξ^α of the fluid particle from its Keplerian circular orbit. One can generalize this equation for the general case of time dependent wave-vector. Such type deviation equations are well-known for the motion of particles in gravitation fields

$$D_\tau^2 \xi^\alpha = -R_{\beta\gamma\delta}^\alpha d_\tau x^\beta \xi^\gamma d_\tau x^\delta, \quad (2.101)$$

where $d_\tau x^\beta$ is the velocity and $R_{\beta\gamma\delta}^\alpha$ is the Riemann tensor; see for example, Eq. (1.8') from the monograph [Misner et al.(1973)]. The secular equation related to deviation equation are basic tools for investigation of the mechanical stability introduced by Lyapunov. In our case the secular equation related to Eq. (2.100) is reduced to the MRI equation Eq. (2.95). In such a way we checked the important test that our general MHD equations give as a particular case MRI predicted by [Velikhov (1959)] and the mechanical model advocated in astrophysics by Balbus and Hawley. Lyapunov analysis of general MHD equations is the simplest way to rederive the MRI.

Here we wish to analyze the differences in SMW amplification. In Keplerian case $\omega_c = -\frac{2}{3}$. The MRI arises by the change of the sign of the spring rigidity ($\omega_A^2 + 2A\Omega$) from the second equation Eq. (2.100), while for the case of pure shear $\omega_c = 0$ we have analogous interval with negative “rigidity” $\left\{ \tilde{K}^2 - 1/[(1 + \xi^2)^2] \right\}$ in Eq. (2.21).

Qualitatively we suppose that strong wave amplification can lead to wave turbulence when we have self-consistent spectral density of magnetosonic waves. We have something like phase transition in an infinite system. Wave turbulence is analogous to the fluctuations of the order parameter. At least on intuitive level this analogy already presents in the theory of the disks; in the review by [Balbus and Hawley (1998)] the word “fluctuations” can be found 49 times. The excess viscosity is actually the order parameter of the wave turbulent phase. In the laminar phase for small Reynolds numbers

$$R \equiv \frac{1}{\nu'_k} = \frac{\Lambda V_A}{\nu_k} = \frac{V_A^2}{A \nu_k} < R_c, \quad (2.102)$$

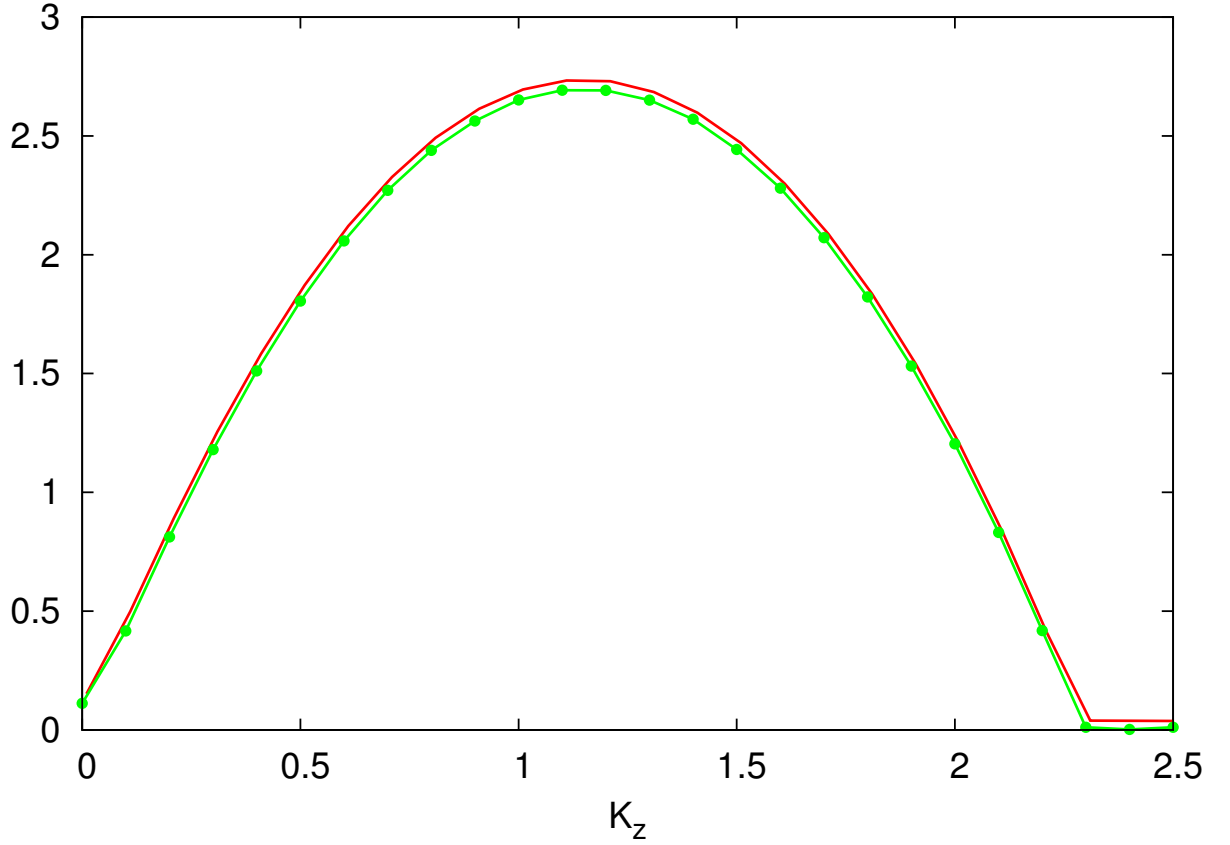


Figure 2.4: The connection between MRI growth rate and giant amplification of SMW revealed by WKB relation Eq. (2.96) for $K_y = 0.01$. Dotted line is the logarithm of the wave gain coefficient $\ln[G = w(\infty)/w(-\infty)]$ multiplied by K_y component of wave vector. Thick line represent the WKB amplification factor during the growth phase [Balbus and Hawley Part IV (1992)]. The WKB formula for $\mu(Q_y, Q_z)$ Eq. (2.96) is asymptotically exact for $Q_y \rightarrow 0$.

the excess viscosity is just zero. The region of Reynolds numbers slightly above the critical one R_c can be investigated by standard numerical methods and already self-sustained turbulence is at least qualitatively confirmed. Now let us speculate the derived results and perspectives.

2.8 Approximate solution

$$\ddot{b}_x + \frac{2\tau Q_y^2(1+\omega_c)}{Q_y^2(1+\tau^2) + Q_z^2} \dot{b}_x + \left[Q_\alpha^2 + 2\omega_c \left(1 - \frac{\tau^2 Q_y^2}{Q_y^2(1+\tau^2) + Q_z^2} \right) \right] b_x = 2\omega_c \left(1 - \frac{\tau^2 Q_y^2}{Q_y^2(1+\tau^2) + Q_z^2} \right) \dot{b}_y$$

$$\ddot{b}_y - \frac{2\tau Q_y^2 \omega_c}{Q_y^2(1+\tau^2) + Q_z^2} \dot{b}_x + Q_\alpha^2 b_y = -\frac{2\tau Q_y^2 \omega_c}{Q_y^2(1+\tau^2) + Q_z^2} b_x + 2 \left[\frac{(\omega_c + 1) Q_y^2}{Q_y^2(1+\tau^2) + Q_z^2} - \omega_c \right] \dot{b}_x$$

$$b_x(\tau) =$$

$$C_1 \text{HeunC}\left(0, -\frac{1}{2}, \omega_c, -\frac{1}{4} \frac{(Q_y^2 + Q_z^2) Q_\alpha}{Q_y^2}, \frac{1}{4} \frac{(Q_\alpha^2 + \omega_c + 2) Q_y^2 + Q_\alpha^2 Q_z}{Q_y^2}, \frac{-\tau Q_y^2}{Q_y^2 + Q_z^2}\right)$$

$$+ C_2 \tau \text{HeunC}\left(0, -\frac{1}{2}, \omega_c, -\frac{1}{4} \frac{(Q_y^2 + Q_z^2) Q_\alpha}{Q_y^2}, \frac{1}{4} \frac{(Q_\alpha^2 + \omega_c + 2) Q_y^2 + Q_\alpha^2 Q_z}{Q_y^2}, \frac{-\tau Q_y^2}{Q_y^2 + Q_z^2}\right)$$

$$b_y(\tau) =$$

$$C_3 \text{HeunC}\left(0, -\frac{1}{2}, 1 + \omega_c, -\frac{1}{4} \frac{(Q_y^2 + Q_z^2) Q_\alpha}{Q_y^2}, \frac{1}{4} \frac{(Q_\alpha^2 + \omega_c + 2) Q_y^2 + Q_\alpha^2 Q_z}{Q_y^2}, \frac{-\tau Q_y^2}{Q_y^2 + Q_z^2}\right) \cdot Q^{2(1+\omega_c)}$$

$$+ C_4 \tau \text{HeunC}\left(0, \frac{1}{2}, 1 + \omega_c, -\frac{1}{4} \frac{(Q_y^2 + Q_z^2) Q_\alpha}{Q_y^2}, \frac{1}{4} \frac{(Q_\alpha^2 + \omega_c + 2) Q_y^2 + Q_\alpha^2 Q_z}{Q_y^2}, \frac{-\tau Q_y^2}{Q_y^2 + Q_z^2}\right) \cdot Q^{2(1+\omega_c)}$$

$$+ 2\omega_c (Q_y^2 + Q_z^2) [Q_y^2(1+\tau^2) + Q_z^2]^{\omega_c} \cdot J$$

(2.103)

$$\ddot{\xi}_r - 2\Omega \dot{\xi}_\varphi = -\omega_A^2 \xi_r,$$

$$\ddot{\xi}_\varphi + 2\Omega \dot{\xi}_r = -(\omega_A^2 + 2A\Omega) \xi_\varphi.$$

$$b_x(\tau) = C_1 e^{w^+ \tau} + C_2 e^{w^- \tau} + C_3 e^{-w^+ \tau} + C_4 e^{-w^- \tau}$$

$$b_y(\tau) = \frac{-1}{2\omega_c Q_\alpha^2} \left[\frac{-w^-}{2} (C_2 e^{w^- \tau} - C_1 e^{-w^- \tau}) (4\omega_c^2 + (w^-)^2 + Q_\alpha^2 + 2\omega_c) \right.$$

$$\left. + \frac{-w^+}{2} (C_3 e^{w^+ \tau} - C_4 e^{-w^+ \tau}) (4\omega_c^2 + (w^+)^2 + Q_\alpha^2 + 2\omega_c) \right]$$

$$w^\pm \equiv \sqrt{-Q_\alpha^2 - 2\omega_c^2 - \omega_c \pm \tilde{w}}, \quad \tilde{w} \equiv |\omega_c| \sqrt{4Q_\alpha^2 + 4\omega_c(\omega_c + 1) + 1}$$

$$\hat{w}^\pm \equiv i\sqrt{Q_\alpha^2 + 2\omega_c^2 + \omega_c \mp \tilde{w}}, \quad \bar{w}^\pm \equiv \hat{w}^\pm (Q_\alpha^2 + 4\omega_c^2 + 2\omega_c + (w^\pm)^2)$$

$$\begin{aligned} \psi(\tau \rightarrow -\infty) &\approx \mathbb{1}(\bar{w}^- \exp(i\hat{w}^- \tau) + \bar{w}^+ \exp(i\hat{w}^+ \tau)) \\ &+ R(\bar{w}^- \exp(-i\hat{w}^- \tau) + \bar{w}^+ \exp(-i\hat{w}^+ \tau), \\ \psi(\tau \rightarrow +\infty) &\approx T(\bar{w}^- \exp(i\hat{w}^- \tau) + \bar{w}^+ \exp(i\hat{w}^+ \tau)). \end{aligned}$$

Using the asymptotics of the eigen-functions

$$\psi_g \approx \begin{cases} D_g[\bar{w}^+ \cos(\hat{w}^+ \tau - \delta_g) + \bar{w}^- \cos(\hat{w}^- \tau - \delta_g)] & \text{for } \tau \rightarrow -\infty, \\ D_g[\bar{w}^+ \cos(\hat{w}^+ \tau + \delta_g) + \bar{w}^- \cos(\hat{w}^- \tau + \delta_g)] & \text{for } \tau \rightarrow +\infty, \end{cases}$$

and

$$\psi_u \approx \begin{cases} -D_u[\bar{w}^+ \cos(\hat{w}^+ \tau - \delta_u) + \bar{w}^- \cos(\hat{w}^- \tau - \delta_u)] & \text{for } \tau \rightarrow -\infty, \\ D_u[\bar{w}^+ \cos(\hat{w}^+ \tau + \delta_u) + \bar{w}^- \cos(\hat{w}^- \tau + \delta_u)] & \text{for } \tau \rightarrow +\infty \end{cases}$$

as well as the general condition

$$\psi(\xi) = C_g^{(q)} \psi_g(\xi) + C_u^{(q)} \psi_u(\xi),$$

For the tunneling coefficient we obtain

$$\mathcal{D} = |T|^2 = s_{ug}^2, \quad s_{ug} \equiv \sin(\delta_u - \delta_g).$$

$$\psi \approx \begin{cases} \bar{w}^+ \cos(\hat{w}^+ \tau - \phi_i) + \bar{w}^- \cos(\hat{w}^- \tau - \phi_i) & \text{for } \tau \rightarrow -\infty, \\ D_f[\bar{w}^+ \cos(\hat{w}^+ \tau + \phi_f) + \bar{w}^- \cos(\hat{w}^- \tau + \phi_f)] & \text{for } \tau \rightarrow +\infty, \end{cases}$$

$$\phi_f = F(\phi_i) \equiv \arctan \frac{s_{ig}s_u + s_{iu}s_g}{s_{ig}c_u + s_{iu}c_g},$$

$$\begin{aligned} s_g &= \sin(\delta_g), & c_g &= \cos(\delta_g), & s_{ig} &\equiv \sin(\phi_i - \delta_g), \\ s_u &= \sin(\delta_u), & c_u &= \cos(\delta_u), & s_{iu} &\equiv \sin(\phi_i - \delta_u) \end{aligned}$$

2.9 Discussion and perspectives

We consider that relation between wave amplification and MRI growth rate in WKB approximation is one important property of the linearized MHD equations. This relation demonstrates that exponential big amplification is a precursor of the MRI and this is a subject of an indisputable mathematics. Let us now clarify what is common and what is different between swing SMW amplification and MRI.

Both phenomena are based on the negative square of the frequency of an effective oscillator which means negative rigidity of the effective spring if we look from eigenvector coordinates. For SMW negative rigidity arises for negative values $\tilde{K}^2 - 1/(1 + \xi^2)^2$ in Eq. (2.21). While for MRI we have negative squares of the frequency in the solution Eq. (2.95) of the bi-quadratic equation Eq. (2.96).

The differences are related to the different physical sense of the effective time. For the SMW the immanent time $\xi = -K_x/\sqrt{K_y^2 + K_z^2} = \tau K_y/\sqrt{K_y^2 + K_z^2}$ has kinematic sense in the wave-vector space, while for MRI we have simply $\tau = At$. As $K_x(t) = -K_y At$ the SMW time ξ has sense only for $K_y \neq 0$ whilst the effective time of MRI strictly speaking $\tau = At$ is applicable only for $K_y = 0$. In this sense SMW amplification and MRI are complimentary.

The different physical nature of the effective times leads also to different mathematical approximations for long wavelength approximation. For SMW without rotation we have δ -function approximation Eq. (2.29) and some people call this over-reflection [Gogoberidze et al. (2004)] as *swing amplification* [Fan and Lou (1997)] caused by a *sudden change* of the rigidity of an effective oscillator. On the other hand for MRI works WKB approximation [Landau and Lifshitz (1981)] and [Migdal (2000)] for the wave amplification Eq. (2.96).

The exponential growth function

$$\mu(Q_y, Q_z) = 2 \int \lambda_{3D}(Q_x, Q_y, Q_z) dQ_x \quad (2.104)$$

have finite limit for $Q_y = 0$, i.e. $\mu(Q_z) = \mu(0, Q_z)$ describes very well the exponentially big wave amplifications $G \approx \exp(\mu(Q_z)/|Q_y|)$ for small $|Q_y| \ll 1$, as it is depicted at Fig. 4. This exponential growth with many decades of amplitude was the motivation for the change of the terminology: the exponentially big wave amplification is called simply MRI because the nonlinear effects can become important before the applicability of the linearized analysis. From formal point of view however the divergence of $G(Q_y \rightarrow 0)$ means that big SMW amplification is the precursor of MRI. We wish to emphasize that for infinite times we have asymptotic wave behavior and this is the reason to speak about the wave amplification. For big enough $|Q_x| \gg 1$ the waves are in good approximation independent and in this sense we have a Kraichnan like MHD-wave turbulence. The wave independence at big wave vectors is analogous to the asymptotic freedom in the elementary particle physics.

Once again we wish to emphasize the distinction between the “SMW swing amplification” and MRI. The SMW swing amplification discovered in Ref. [Chagelishvili et al.(1993)] occurs even without rotation for $\omega = 0$. This amplification is divergent for $Q_\perp = \sqrt{Q_y^2 + Q_z^2} = 0$. This is only a point in (Q_y, Q_z) plane, let us call this point Γ .

MRI however arises only for *anticyclonic alignment* of the angular speed Ω and rotation of shear velocity $\omega = \Omega \cdot \text{rot} \mathbf{V}_{\text{shear}} / A^2 < 0$. The dimensionless angular velocity ω is analogous to the reduced temperature $\epsilon = (T - T_c) / T_c$ from the physics of the phase transitions and critical phenomena [Patashinski and Pokrovskii (1979)]. For evanescent negative ω MRI arises close to the Γ -point, i.e. infinite amplification related to MRI arises for $\omega = -0$ at the point where swing SMW amplification is divergent. In this sense swing SMW amplification is the precursor of the MRI. Analogously divergent susceptibility at $\epsilon = +0$ is the precursor of the appearance of the order parameter at $\epsilon = -0$. Below the critical temperature in the physics of second order phase transitions we have order parameter while for our MHD system we have MRI. For Keplerian value of the angular velocity $\omega_{\text{Kepler}} = -\frac{2}{3}$ the initial Γ -point of divergent amplification is extended to a finite interval in Q_z axis with $Q_y = 0$, see Fig. 4. That is why we can say that divergent-swing-SMW amplification for $\omega = 0$ is the precursor of the divergent MRI amplification for negative anticyclonic $\omega < 0$.

As we already cited [Balbus and Hawley (1998)] there is already an emerging consensus amidst the astrophysicist that MRI is the heart of the disk turbulence. Here we wish to repeat the terminology. Strictly speaking instability means exponential amplification which occurs only in a manifold with measure zero $Q_y = 0$. As poetic metaphor we can use the notion instability (I) if some mode have a time interval with almost exponential growth which is described in an acceptable accuracy with WKB approximation; in our case Eq. (2.104) This instability leads to giant amplification of MHD waves which also can be call heart of the turbulence. However our analytical solution has revealed that divergent SMW amplification occurs even in the case of pure shear without rotation ω_c . Generalizing one can say that heart continues even without rotation and the reason of the self-sustained turbulence is the divergent wave amplification at small wave-vectors. Due to drift velocity on the momentum space Eq. (10) and the small bare viscosity the result of wave amplification and MRI propagates along $Q_x \mathbf{e}_x$ direction at very big wave-numbers.

WAVE TURBULENCE

3.1 Incorporation of Turbulence as Random Driver of MHD Waves

Being as efficient as has been demonstrated above, the amplification of SMWs, Eq. (2.81), is possibly the dominant physical factor responsible for generating and maintaining the turbulence in accretion flows. However the theory of turbulence is much more complicated than the theory of linearized waves. That is why here we provide an illustration how the wave amplification can be incorporated in the turbulence theory. In order to establish common set of notions and notations we will recall some basic properties of the homogeneous isotropic Kolmogorov–Obukhov turbulence.

3.1.1 Kolmogorov Turbulence

Let the velocity be presented by the Fourier integral

$$\mathbf{V}(\mathbf{r}) = \int e^{i\mathbf{k}\cdot\mathbf{r}} \mathbf{V}_{\mathbf{k}} \frac{d^3k}{(2\pi)^3}, \quad \mathbf{V}_{\mathbf{k}} = \int \mathbf{V}(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} d^3x. \quad (3.1)$$

The energy per unit mass is

$$\int \frac{1}{2} V^2(\mathbf{r}) d^3x = \int \frac{1}{2} V_{\mathbf{k}}^2 \frac{d^3k}{(2\pi)^3} = \int \mathcal{E}_{\mathbf{k}} \frac{d^3k}{(2\pi)^3}, \quad (3.2)$$

where we introduce the spectral density averaged with respect to the turbulence

$$\mathcal{E}_{\mathbf{k}} = \frac{1}{2} \langle V_{\mathbf{k}}^2 \rangle_{\text{turbulence}}. \quad (3.3)$$

We can introduce also the energy of vortices which contains Fourier components with wavelength $2\pi/k$, shorter than some fixed length λ

$$\frac{1}{2}V_\lambda^2 = \int_{k < 1/\lambda} \mathcal{E}_k \frac{d^3k}{(2\pi)^3}, \quad (3.4)$$

where for isotropic turbulence $\mathcal{E}_k = \mathcal{E}_k$. This energy evaluates turbulent pulsation with size λ . Further on we will continue with only order of magnitude evaluations, hence in the following estimations we will drop off factors such as 4π , $\frac{1}{2}$, etc.

According to the Kolmogorov–Obukhov (KO) scenario in the inertial range the magnitude of the velocity pulsations V_λ can depend only on the turbulent power dissipated per unit mass ε . There is only one combination with the appropriate dimension

$$\varepsilon = \frac{V_\lambda^2}{\lambda/V_\lambda} = \frac{\text{energy/mass}}{\text{time} = \text{length/velocity}} = \frac{\text{power}}{\text{mass}}, \quad (3.5)$$

which yields

$$\begin{aligned} V_\lambda^2 &\sim (\varepsilon\lambda)^{2/3} \sim \int_{k\lambda > 1} \mathcal{E}_k^{\text{KO}} d^3k \sim \int_{k\lambda > 1} \frac{\varepsilon^{2/3}}{k^{5/3}} dk, & d^3k &\sim k^2 dk, \\ E(k) &= \int k^2 \mathcal{E}_k^{\text{KO}} d\Omega \approx C_K \varepsilon^{2/3} k^{-5/3}, & C_K &\approx 1.6, & \mathcal{E}_k^{\text{KO}} &\sim \frac{\varepsilon^{2/3}}{k^{11/3}}. \end{aligned} \quad (3.6)$$

V_λ is the amplitude of variation in the velocity pulsation at distance λ . \mathcal{E}_k is the energy density in the \mathbf{k} -space per unit mass; in the Kolmogorov–Obukhov picture this is a static variable.

The scaling law $V_\lambda \sim (\varepsilon\lambda)^{1/3}$ Eq. (3.5) is applicable for large enough distances $\lambda > \lambda_0$, where λ_0 describes the scale where dissipation effects become essential.

Let us now consider a magnetosonic wave with a time-dependent wave-vector

$$q_x(t) = -A(t - t_0)q_y, \quad q_y = \text{const}, \quad q_z = \text{const}, \quad (3.7)$$

and time-dependent energy density per unit mass in real space

$$w(t) = \frac{1}{2} \langle \mathbf{V}_{\text{wave}}^2 + \frac{\mathbf{B}_{\text{wave}}^2}{\rho\mu_0} \rangle = \frac{V_A^2}{4} [\mathbf{b}^2 + (d_{Q\xi}\mathbf{b})^2], \quad (3.8)$$

where $\langle \dots \rangle$ stands for spatial averaging, $\langle \cos^2(\mathbf{k} \cdot \mathbf{r}) \rangle = \frac{1}{2}$. Then the energy density in the \mathbf{k} -space is

$$\mathcal{E}_k(t) = w(t) \delta[\mathbf{k} - \mathbf{q}(t)]. \quad (3.9)$$

Let us mention that all MHD variables \mathbf{b} , \mathbf{v} , w in Eq. (3.8), and P depend on the effective wave functions ψ and χ (solutions to the effective Schrödinger equations) through the dimensionless time ξ . Having in the beginning $t = t_0$ a distribution of the magnetic field $\mathbf{B}_{\text{wave}}(\mathbf{r}, t_0)$ and

velocity with $\nabla \cdot \mathbf{V}_{\text{wave}}(\mathbf{r}, t_0) = 0$, we can calculate the Fourier components

$$\mathbf{v}_{\mathbf{k}}(t_0) = i \int \frac{\mathbf{V}_{\text{wave}}(\mathbf{r}, t_0)}{V_A} e^{-i\mathbf{k} \cdot \mathbf{r}} d^3x, \quad (3.10)$$

$$\mathbf{b}_{\mathbf{k}}(t_0) = \int \frac{\mathbf{B}_{\text{wave}}(\mathbf{r}, t_0)}{B_0} e^{-i\mathbf{k} \cdot \mathbf{r}} d^3x, \quad (3.11)$$

$$\xi_{0,\mathbf{k}} \equiv -\frac{k_x}{\sqrt{k_y^2 + k_z^2}}, \quad (3.12)$$

and initial dimensionless time $\xi_{0,\mathbf{k}}$. If $k_z = 0$ then $\text{sgn}(k_y)\xi_{0,\mathbf{k}} = \tau_{0,\mathbf{k}} = -k_x/k_y$. Then we have to determine the coefficients C in the general solutions for ψ and χ using the initial values at t_0

$$\mathbf{b}_{\mathbf{k}}(\xi) = C_g \mathbf{b}_g + C_u \mathbf{b}_u + \tilde{C}_g \mathbf{b}_{\tilde{g}} + \tilde{C}_u \mathbf{b}_{\tilde{u}}, \quad (3.13)$$

$$\mathbf{v}_{\mathbf{k}}(\xi) = C_g \mathbf{v}_g + C_v \mathbf{v}_u + \tilde{C}_g \mathbf{v}_{\tilde{g}} + \tilde{C}_u \mathbf{v}_{\tilde{u}}, \quad (3.14)$$

$$\mathbf{k} \cdot \mathbf{b}_{\mathbf{k}} = 0 = \mathbf{k} \cdot \mathbf{v}_{\mathbf{k}}. \quad (3.15)$$

In this set we can use only x - and z -components, and so we obtain 4 equation for the constants C_g , C_u , \tilde{C}_g , and \tilde{C}_u . The functions $\mathbf{b}_g(\xi)$ and $\mathbf{v}_g(\xi)$ are defined via substituting ψ_g , and analogously ψ_u , χ_g , and χ_u . Then at each moment t we can calculate all variables in the \mathbf{k} -space

$$\xi_{\mathbf{k}}(t) = \xi_{0,\mathbf{k}} + (t - t_0) A \frac{k_y}{\sqrt{k_y^2 + k_z^2}}, \quad (3.16)$$

$$\mathbf{b}_{\mathbf{k}}(t) = \mathbf{b}_{\mathbf{k}}(\xi_{\mathbf{k}}(t)), \quad (3.17)$$

$$\mathbf{v}_{\mathbf{k}}(t) = \mathbf{v}_{\mathbf{k}}(\xi_{\mathbf{k}}(t)), \quad (3.18)$$

$$k_x^{\text{wave}}(t) = k_x - (t - t_0) A k_y. \quad (3.19)$$

Finally, we can return back to the real \mathbf{r} -space

$$\begin{aligned} \mathbf{V}_{\text{wave}}(\mathbf{r}, t) &= -iV_A \int \mathbf{v}_{\mathbf{k}}(t) e^{-i[\mathbf{k} \cdot \mathbf{r} - (t-t_0) A k_y x]} \frac{d^3k}{(2\pi)^3}, \\ \mathbf{B}_{\text{wave}}(\mathbf{r}, t) &= B_0 \int \mathbf{b}_{\mathbf{k}}(t) e^{-i[\mathbf{k} \cdot \mathbf{r} - (t-t_0) A k_y x]} \frac{d^3k}{(2\pi)^3}, \\ \text{Re}(e^{-i\varphi}) &= \cos \varphi, \quad \text{Re}(-ie^{-i\varphi}) = -\sin \varphi. \end{aligned} \quad (3.20)$$

This evolution of MHD variables is the main detail of the theory of MHD turbulence in a shear flow.

Consider now an imaginary fluid filling the phase space \mathbf{k} and $w(t) \equiv \mathcal{E}_{\mathbf{k}}(t)$ from Eq. (3.9) being the energy density carried by a droplet of that fluid. As a wave mode initially with wave-vector \mathbf{k} evolves according to Eq. (3.7), the infinitesimal phase-fluid droplet associated with that mode moves in the \mathbf{k} -space. Wave amplification means that the energy density of the droplets increases by a factor of G

$$G = \frac{w(t \rightarrow +\infty)}{w(t \rightarrow -\infty)}. \quad (3.21)$$

Indeed for $\chi = 0$, $k_z = 0$, and $\xi \rightarrow \infty$

$$b_y^2 \asymp \psi^2 \gg b_x^2 + b_z^2, \quad v_y^2 \asymp \left(\frac{d_\xi \psi}{Q} \right)^2 \gg v_x^2 + v_z^2 \quad (3.22)$$

and

$$w(t \rightarrow \infty) = \frac{1}{4} V_A^2 D_f^2, \quad w(t \rightarrow -\infty) = \frac{1}{4} V_A^2. \quad (3.23)$$

For big enough time arguments $|\xi| \gg 1$ and purely two-dimensional waves with $k_z = 0$ the motion of the fluid asymptotically corresponds to a SMW with dispersion coinciding with the AW one

$$Q\xi = \omega_{\text{SMW}} t, \quad \omega_{\text{SMW}} = \omega_{\text{AW}} = V_A |k_y|, \\ \psi(\xi) \asymp D_f \cos(\omega_{\text{SMW}} t + \phi_f). \quad (3.24)$$

The Poynting vector, i.e., the energy flux in **r**-space is $V_A w$.

The velocity of the droplet in the **k**-space according to Eq. (3.7) determines the field of the shear flow in the **k**-space

$$\mathbf{U} = d_t \mathbf{q}(t) = -A q_y \mathbf{e}_x, \quad \mathbf{U}_{\mathbf{k}}^{\text{shear}} = -A k_y \mathbf{e}_x. \quad (3.25)$$

Looking at a droplet we actually derive the shear flow velocity field in **k**-space, $\mathbf{U}_{\mathbf{k}}^{\text{shear}}$.

According to the Kolmogorov–Obukhov cascade of energy we have a constant energy flux through each spherical surface with surface element df in **k**-space

$$\varepsilon = \oint \mathcal{E}_{\mathbf{k}}^{\text{KO}} \mathbf{U}_{\mathbf{k}}^{\text{KO}} d\mathbf{f} = \mathcal{E}_{\mathbf{k}}^{\text{KO}} U_{\mathbf{k}}^{\text{KO}} 4\pi k^2, \quad (3.26)$$

which gives

$$\mathbf{U}_{\mathbf{k}}^{\text{KO}} \sim \varepsilon^{1/3} k^{5/3} \mathbf{e}_k, \quad \mathbf{e}_k = \frac{\mathbf{k}}{k}, \quad (3.27)$$

i.e., the velocity in **k**-space has dimension $1/(\text{time} \times \text{length})$. Here we used an important for our further work notion of the energy flux in the **k**-space

$$\mathbf{S} = \mathcal{E}_{\mathbf{k}} \mathbf{U}_{\mathbf{k}} \quad (3.28)$$

which is equal to energy density times velocity in the **k**-space. This notion is analogous to the Poynting vector being, however, defined in the **k**-space. In the Kolmogorov–Obukhov scenario we have

$$\frac{\partial}{\partial \mathbf{k}} \cdot \mathbf{S}^{\text{KO}} = \varepsilon \delta(\mathbf{k}). \quad (3.29)$$

In order to approximate the turbulence as an initial source of MHD waves we have to merge the turbulence with the wave spectral densities and velocities. The simplest possible scenario is given in the next subsection.

3.1.2 Derivation of Shakura–Sunyaev Phenomenology in the Framework of Kolmogorov Turbulence

How vortices create waves is a complicated problem far beyond the scope of the present study. Here we will give only a model illustration merging the spectral density of vortices $\mathcal{E}_k^{\text{turb}}$ from Kolmogorov turbulence with spectral density of magnetosonic waves $\mathcal{E}_k^{\text{wave}}$

$$\mathcal{E}_k^{\text{wave}} \sim \mathcal{E}_k^{\text{turb}} \sim \mathcal{E}_\Lambda = \varepsilon^{2/3} \Lambda^{11/3}, \quad (3.30)$$

on the plane in momentum space

$$k_x = -\text{sgn}(k_y) \Lambda^{-1}, \quad k_y^2 + k_z^2 < \Lambda^{-2}, \quad (3.31)$$

where we qualitatively suppose that vortices are converted into waves. Sign function corresponds to the direction of the shear flow in the \mathbf{k} -space, Eq. (3.25). For k_y we consider that turbulent vortices have a given spectral density at $k_x > \Lambda^{-1}$ which is converted to MHD wave energy at the plane $k_x = \Lambda^{-1}$, and further on this wave energy evolves according to our solution. In other words, the plane $k_x > \Lambda^{-1}$ is the boundary between the vortex region and the beginning of the amplification in the wave region where vortices have negligible influence. In our qualitative picture we suppose that vortices create spectral density which further on evolves as wave spectral density with negligible influence.

The amplification Eq. (2.81) is essential $\mathcal{G} \gg 1$ only within a cylinder

$$\mathcal{G}(k_y, k_z) - 1 \sim \frac{1}{\Lambda^2 q^2}, \quad q^2 = k_y^2 + k_z^2 < \Lambda^{-2} \quad (3.32)$$

with radius Λ^{-1} . This result with remains unchanged in amplitude if we include the $J_{c,u}$ and $J_{s,g}$ terms.

The amplification occurs in the region $-\Lambda^{-1} < k_x < \Lambda^{-1}$, that is to say from the cylinder we cut a tube with length $2\Lambda^{-1}$. In other words, we have a domain with a shape of a tube in momentum space

$$\mathcal{V} = \{k_y^2 + k_z^2 < \Lambda^{-2}, \quad |k_x| < \Lambda^{-1}\}. \quad (3.33)$$

In order to calculate the total power of waves \mathcal{H} (per unit mass) analogously to Eq. (3.26) we will integrate the energy flux on the surface of the tube

$$\mathcal{H} = \oint \mathcal{E}_k^{\text{wave}} \mathbf{U}_k^{\text{shear}} d\mathbf{f}. \quad (3.34)$$

As the shear in the physical flow results in a drift of the wave modes along the axis of the tube, we have to take into account only the circular surfaces

$$\epsilon \sim \int |U_x| [\mathcal{G}(k_y, k_z) - 1] \mathcal{E}_\Lambda dk_y dk_z. \quad (3.35)$$

The multiplier $(\mathcal{G} - 1)$ takes into account the difference between flowing out and flowing in energy fluxes.

We can use polar coordinates

$$k_z = q \cos \theta, \quad k_y = q \sin \theta, \quad U_x = Aq \sin \theta. \quad (3.36)$$

Averaging over the angle θ

$$\langle U_x \rangle = \frac{2}{\pi} Aq \sim Aq, \quad \int_0^\pi \sin \theta \frac{d\theta}{\pi} = \frac{2}{\pi}, \quad (3.37)$$

and substituting it in Eq. (3.34), using $dk_y dk_z = d(\pi q^2)$, leads to the simple integral

$$\mathcal{H} \sim \int_0^{\Lambda^{-1}} \frac{Aq}{\Lambda^2 q^2} \mathcal{E}_\Lambda q dq \sim \frac{A \mathcal{E}_\Lambda}{\Lambda^3} \sim AV_\Lambda^2. \quad (3.38)$$

Then for the volume density of the amplified waves we have

$$\mathcal{Q} \equiv \rho \mathcal{H} \sim \rho AV_\Lambda^2 \sim \rho (\varepsilon V_A)^{2/3} A^{1/3}. \quad (3.39)$$

As all waves are finally dissipated, \mathcal{Q} is actually the volume density of plasma heating.

For evanescent Kolmogorov turbulence power

$$\rho \varepsilon \ll A \rho V_A^2 = AB_0^2 / \mu_0 \quad (3.40)$$

the heating power \mathcal{H} has a critical behavior

$$d_\varepsilon \mathcal{H} \sim G_{\text{turb}} \equiv \frac{\mathcal{H}}{\varepsilon} \sim \left(\frac{AV_A^2}{\varepsilon} \right)^{1/3} \gg 1, \quad \mathcal{H} \gg \varepsilon \quad \text{for } \varepsilon \rightarrow 0 \quad (3.41)$$

which demonstrates that disks can ignite as a star even for very weak turbulence and magnetic field. The ratio of wave power and Kolmogorov vortex power G_{turb} can be considered as an amplification coefficient for the turbulence. This energy gain shows how efficient is the transformation of shear flow energy into waves or in a broader framework the transformation of gravitational energy into heat of accretion disks.

For hydrogen plasma $\rho c_s^2 / p = 5/3 \sim 1$. Now we can evaluate the shear stress (as given by the ratio of the volume density of heating power and the shear frequency)

$$\sigma = \frac{2\mathcal{Q}}{A} \sim \rho \left(\frac{\varepsilon V_A}{A} \right)^{2/3} \sim \rho V_\Lambda^2 \quad (3.42)$$

via an effective viscosity

$$\eta_{\text{eff}} = \frac{\sigma}{A} \sim \rho \frac{(\varepsilon V_A)^{2/3}}{A^{5/3}}, \quad \nu_{\text{eff}} = \frac{\eta_{\text{eff}}}{\rho} \sim \frac{\mathcal{H}}{A^2} \sim \frac{(\varepsilon V_A)^{2/3}}{A^{5/3}} \quad (3.43)$$

and the dimensionless Shakura–Sunyaev friction coefficient

$$\alpha \equiv \frac{\sigma}{p} \sim \frac{V_\Lambda^2}{c_s^2}. \quad (3.44)$$

Including of the energy of z -polarized AWs does not modify this result. Here we wish to emphasize that in our evaluation of the energy gain, we were concentrated on the wave amplification of the energy of two dimensional motion in the x - y plane. Taking into account the energy in z -direction shows that the heating is even higher, which is of course in the favor of the concepts.

For an approximately Keplerian disk rotation the shear rate is half of the frequency of the orbital Keplerian angular velocity $A = -\frac{1}{2}\omega_{\text{Kepler}}$. In this case for time A^{-1} the disk rotates per 2 radians. For Earth's rotation along the Sun this time is of the order of one season. In such a way the length parameter of our problem $\Lambda = V_A/A$ can be evaluated as one Alfvén season. Then V_A from the final result for the Shakura–Sunyaev parameter can be qualitatively considered as a pulsation of the turbulent velocity for two disk particles at distance equal to one Alfvén season Λ . Our theory is formally applicable for $V_A \ll c_s$ but the boundary of its applicability (when compressibility effects stop the SMWs amplification) allows us to understand that strong disk's turbulence can lead to Shakura–Sunyaev upper limit $\alpha \sim 1$. Thus the following cascade of events emerges as a likely scenario for the intense heating in accretion flows: the heating of the bulk of the disk creates convection. For strong heating the convection is turbulent. Turbulence generates magnetohydrodynamic waves. Waves are amplified by the shear flow – this is the transformation of gravitational energy of orbiting plasma into waves. Waves finally are absorbed by the viscosity which creates the heating. The heat is emitted through the surface of the disk. This process of formation of stars and other compact astrophysical objects from nebulae works continuously – we have a self-consistent theory for self-sustained turbulence of the magnetized accretion disks.

The weak point of this scenario is the supposed convective turbulence which in presence of magnetic fields is unlikely to be of Kolmogorov type. We consider as much more plausible scenario the appearance of a self-sustained magneto-hydrodynamical turbulence considered in the next subsection.

3.1.3 Kraichnan Turbulence as a more Plausible Scenario for Accretion Disks

Magnetic field qualitatively changes the behavior of the fluid. We have no waves generated by vortices – the turbulence in magnetic field is related to MHD waves. Analogously to the Kolmogorov law Eq. (3.5), for the Kraichnan turbulence the power of energy cascade in the dissipation-free regime is given by the wave–wave interaction

$$\varepsilon = \frac{(V_\lambda^2)^2}{\lambda V_A} = \frac{(\text{velocity})^3}{\text{length}} = \frac{\text{power}}{\text{mass}}. \quad (3.45)$$

This power is proportional to the intensity of the two interacting waves and this nonlinear effect for incompressible fluid is due to the convective term $\mathbf{V} \cdot \nabla \mathbf{V}$ of the substantial acceleration $D_t \mathbf{V} = \partial_t \mathbf{V} + \mathbf{V} \cdot \nabla \mathbf{V}$ of the momentum equation.

The theory of generation of SMWs invokes parallels with the nonlinear optical phenomena in lasers. The velocity oscillations of two amplified MHD waves $\mathbf{V}^{(a)}$ and $\mathbf{V}^{(b)}$ create an external driving force of the new wave with velocity field \mathbf{V} . In the linearized we have to insert a small

nonlinear correction

$$\rho \partial_t \mathbf{V} = -\nabla p + \frac{\nabla \times \mathbf{B}}{\mu_0} \times \mathbf{B} + \rho \mathbf{f}, \quad \mathbf{f} \equiv \frac{1}{2} \sum_{a,b} \mathbf{V}^{(a)} \cdot \nabla \mathbf{V}^{(b)}. \quad (3.46)$$

Here, in the inhomogeneous term \mathbf{f} we have to perform summation over all other MHD waves. This external for the wave force (per unit mass) acts as an external noise and its statistical properties are determined by the force–force correlator

$$\hat{\Gamma}(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) = \langle \mathbf{f}(\mathbf{r}_1, t_1) \mathbf{f}(\mathbf{r}_2, t_2) \rangle, \quad (3.47)$$

where the averaging is over the waves phases. A scenario of such type (a Langevin MHD) was described in Ref. [[Mishonov et al. \(2007\)](#)]; this approach is similar in the spirit to the forced burgers turbulence. [[Woyczyński \(1998\)](#)] In the framework of that scenario the strongly amplified $|D_f| \gg 1$ MHD waves with asymptotics Eq. (2.8)

$$\psi \approx D_f \theta(\xi) \cos(Q\xi + \phi_f) \exp(-\nu' K_y^2 \tau^3 / 3) \quad (3.48)$$

generate new waves and after a statistical averaging we have a self-consistent theory for magnetic turbulence in a shear flow. So MHD waves ignite the chain reaction of quasar self-heating. The last exponential term describes the wave damping when a small viscosity is taken into account. Damping is significant only for $t \rightarrow \infty$ when $|k_x| \gg |k_y|$ and the wave-vector is almost parallel to the magnetic field. In this geometry, the damping rate of the wave density of AWs and SMWs is proportional to the square of the frequency [[Landau and Lifshitz \(1983\)](#)]

$$w(t) = w(0) \exp\left(-\frac{\omega^2}{V_A^2} \nu t\right) = w(0) \exp(-\nu k^2 t). \quad (3.49)$$

For the time-dependent wave-vector $k^2(t) \approx (k_y A t)^2$ in the argument of the exponent we have to make the replacement

$$\nu k^2 t \rightarrow \nu \int_0^t k^2(t') dt' = \frac{1}{3} \nu' K_y^2 \tau^3, \quad \nu' = \frac{\nu A}{V_A^2} = \frac{1}{\mathcal{R}}, \quad \mathcal{R} \equiv \frac{\Lambda V_A}{\nu_k}, \quad \mathcal{S}_{\text{MRI}} \equiv \frac{\Lambda V_A}{\nu_m}, \quad (3.50)$$

where ν' is the dimensionless viscosity and $\mathcal{R} = V_A^2/A$ is the “Reynolds number of the magnetorotational instability (MRI), [[Masada and Sano \(2008\)](#)]” and \mathcal{S}_{MRI} is the Lundquist number of MRI. After long enough time t_ν when

$$|k_x(t_\nu)| = 1/\lambda_a, \quad \lambda_a = \frac{\nu}{V_A}, \quad (3.51)$$

MHD waves are completely dissipated.

The details of self-consistent MHD turbulence will be given elsewhere, but again the wave amplification operates as a turbulence amplifier. For MHD turbulence one can expect

$$\sigma_{R\varphi} = \alpha_m(\nu') p_B, \quad p_B = \frac{1}{2} \rho V_A^2. \quad (3.52)$$

The evaluation of the magnetic friction coefficient α_m as a function of the dimensionless viscosity is a new problem addressed to the theoretical astrophysics.

3.2 Review of used results

Imagine that *with painstaking numerical work* [Balbus and Hawley Part III (1992)] the *frustratingly complex* system of MHD equations Eq. (2.6) and Eq. (2.7) is already solved. Using this solution we can express the effective kinematic viscosity

$$\nu_{\text{eff}}(\tau) = \nu_k + \nu_k \sum_{\mathbf{Q}} Q^2 \mathbf{v}_{\mathbf{Q}}^2(\tau) + \nu_m \sum_{\mathbf{Q}} Q^2 \mathbf{b}_{\mathbf{Q}}^2(\tau). \quad (3.53)$$

The additional time averaged wave viscosity

$$\eta_{\text{wave}} = \lim_{\Delta\tau \rightarrow \infty} \int_0^{\Delta\tau} \frac{d\tau}{\Delta\tau} \left(\rho\nu_k \sum_{\mathbf{Q}} Q^2 \mathbf{v}_{\mathbf{Q}}^2(\tau) + \rho\nu_m \sum_{\mathbf{Q}} Q^2 \mathbf{b}_{\mathbf{Q}}^2(\tau) \right) \quad (3.54)$$

is created by the momentum transfer by the SMW. In such a way the “alfvéons” are the new particles which can have even dominant contribution to the viscosity. We wish to emphasize that this expression is an exact result for used model of incompressible fluid; for compressible fluid expression is more complicated and contains the second viscosity. For the usual hydrodynamics Boussinesq [Boussinesq (1877)] and Prandtl [Prandtl (1925)] pointed out the role of vortices for the momentum transport and creation of the turbulent viscosity. The magnetic field in conducting plasmas opens the new possibility the role of the vortices to be substituted by MHD waves. For accretion disks the problem for the effective viscosity is reduced to calculation of the spectral densities of the wave turbulence presented by Eq. (3.53) and Eq. (3.54).

The MHD equations are investigated in coordinate space in uncountable set of numerical works. However development in this direction is almost saturated and we have to look for other possibilities. Numerical solution in wave-vector space gives a promising perspective especially if significant part of calculation of nonlinear terms is made analytically using analytical asymptotics for large wave-vectors. In this case the MHD modes are almost independent. We have something like asymptotic freedom known from the elementary particles physics. MHD modes are generated and amplified at small wave-vectors where numerical analysis is indispensable. The numerical solution from the finite wave-vector domain has to be continued by the analytical asymptotic which gives the possibility for fast calculation of the nonlinear terms. We believe that further progress can be made by the essential use of the wave-vector space. That is why one of the goals of the present research is to write the general nonlinear equation in wave-vector space and to analyze the properties of the linearized ones. We have to know MHD set of equations which have to be solved.

The parallel between the propagation of the waves and particles was pointed out by Hamilton. We recall this analogy just to emphasize the differences. For the elementary particles of the plasmas the elementary acts of the scattering are in a very good approximation independent and this facilitates the kinetic problem. While MHD turbulence is a hidden coherent structure described by the very difficult for solution system Eq. (2.6) and Eq. (2.7). Instead of separate particles we have something like phase transition in a fluctuating Bose condensate. For example, in the two-dimensional fluid the equilibrium statistics has a most interesting structure, despite the

simplicity of the energy expression, because there is an additional important constant of motion: the enstrophy, or integrated square of the vorticity. The enstrophy constant leads to equilibria in which a large fraction of the energy is condensed into the largest spatial scales of motion, a situation closely analogous to the Bose–Einstein condensation in an ideal boson gas [Kraichnan (1980)].

The simplest possible scenario is that the system have stable static solution, $\mathbf{v}_Q(\tau) = \mathbf{v}_Q$ and $\mathbf{b}_Q(\tau) = \mathbf{b}_Q$, and analogously to the turbulence above the ocean [Miles (1957)] we have again a *cooperative behavior buried in* the wave turbulence. In this case it is not necessary to make time averaging and for the enhancement factor of the viscosity we obtain [Dimitrov et al. (2011)]

$$Z = \frac{\eta_{\text{eff}}}{\eta} = \frac{\sigma_{xy}}{\eta A} = \frac{\tilde{Q}}{\eta A^2} = 1 + \int_Q \left(\mathbf{v}_Q^2 + \frac{\mathbf{b}_Q^2}{P_m} \right) Q^2 \frac{dQ_x dQ_y dQ_z}{(2\pi)^3}, \quad \frac{1}{P_m} = \frac{\varepsilon_0 c^2 \varrho_\Omega}{\eta/\rho}. \quad (3.55)$$

The viscosity renormalization parameter $Z \gg 1$ (for quasars is possible $\ln Z \simeq 20$) determines the work of the accretion disks as a machine for making of compact astrophysical objects; $\sigma_{xy} \equiv \sigma_{r\phi}$ is the time averaged stress tensor and \tilde{Q} is the volume density of heating power. The formulas for the effective viscosity are the tool to take into account the strong anisotropy of the wave turbulence, i.e. the anisotropy of the integrand in the expression above. As we mentioned this result is exact in the framework of MHD for incompressible fluid. Due to general mechanical theorem the time averaged volume density of heating power, shear component of stress tensor and homogeneous shear rate are related with the simple equation $\tilde{Q} = \sigma_{xy} A = \eta_{\text{eff}} A^2$ – this is the definition for effective viscosity η_{eff} determined by Ohmic resistivity, bare viscose friction, and Fourier components of the MHD variables.

The last term in Eq. (3.55) can be essential only in the initial stages of the accretion disk where the hydrogen is weakly ionized. In this case before the re-ionization of the hydrogen after the big bang the magnetic Prandtl number is small $P_m = \nu_k/\nu_m \ll 1$. In this regime of weakly ionized plasma the viscosity coefficient is determined Ohmic dissipation.

The nonlinear hydrodynamic terms are known from the monographs on hydrodynamics [Pope (2000)] and magnetohydrodynamics [Biskamp (2003)]. In our work we use dimensionless imaginary components of the velocity $\mathbf{v}_Q(\tau)$ in order the final MHD system to be real.

Our starting point are general magnetohydrodynamic (MHD) equations in wave-vector space

$$D_{\bar{\tau}}^{\text{shear}} \mathbf{v}_{\mathbf{Q}}(\tau) = -v_{x,\mathbf{Q}} \mathbf{e}_y + 2n_y \mathbf{n} v_{x,\mathbf{Q}} + 2\omega_c \mathbf{n} (n_y v_{x,\mathbf{Q}} - n_x v_{y,\mathbf{Q}}) - 2\omega_c \times \mathbf{v}_{\mathbf{Q}} \quad (3.56)$$

$$+ (\boldsymbol{\alpha} \cdot \mathbf{Q}) \mathbf{b}_{\mathbf{Q}} - \nu'_k Q^2 \mathbf{v}_{\mathbf{Q}} + \mathbf{N}_v, \quad (3.57)$$

$$\mathbf{N}_v = \Pi^{\perp \mathbf{Q}} \cdot \sum_{\mathbf{Q}'} [\mathbf{v}_{\mathbf{Q}'} \otimes \mathbf{v}_{\mathbf{Q}-\mathbf{Q}'} + \mathbf{b}_{\mathbf{Q}'} \otimes \mathbf{b}_{\mathbf{Q}-\mathbf{Q}'}] \cdot \mathbf{Q}, \quad (3.58)$$

$$\mathbf{Q} \cdot \mathbf{N}_v = 0, \quad \mathbf{v}_{\mathbf{Q}}(\bar{\tau}_0) = \Pi^{\perp \mathbf{Q}} \mathbf{v}_{\mathbf{Q}}(\bar{\tau}_0),$$

$$D_{\bar{\tau}}^{\text{shear}} \mathbf{b}_{\mathbf{Q}}(\tau) = b_{x,\mathbf{Q}} \mathbf{e}_y - (\mathbf{Q} \cdot \boldsymbol{\alpha}) \mathbf{v}_{\mathbf{Q}} - \nu'_m Q^2 \mathbf{b}_{\mathbf{Q}} + \mathbf{N}_b, \quad (3.59)$$

$$\mathbf{N}_b = \Pi^{\perp \mathbf{Q}} \cdot \sum_{\mathbf{Q}'} [\mathbf{b}_{\mathbf{Q}'} \otimes \mathbf{v}_{\mathbf{Q}-\mathbf{Q}'} - \mathbf{v}_{\mathbf{Q}'} \otimes \mathbf{b}_{\mathbf{Q}-\mathbf{Q}'}] \cdot \mathbf{Q}, \quad (3.60)$$

$$\mathbf{Q} \cdot \mathbf{N}_b = 0, \quad \mathbf{b}_{\mathbf{Q}}(\bar{\tau}_0) = \Pi^{\perp \mathbf{Q}} \mathbf{b}_{\mathbf{Q}}(\bar{\tau}_0),$$

$$\Pi^{\perp \mathbf{Q}} \equiv \mathbb{1} - \mathbf{n} \otimes \mathbf{n}, \quad \mathbf{n} \equiv \frac{\mathbf{Q}}{Q}, \quad \bar{\tau} \equiv -\frac{Q_x}{Q_y} \quad (3.61)$$

$$D_{\tau}^{\text{shear}} \equiv \partial_{\tau} + \mathbf{U}_{\text{shear}}(\mathbf{Q}) \cdot \partial_{\mathbf{Q}} = \partial_{\tau} - Q_y \partial_{Q_x} = \partial_{\tau} + \partial_{\bar{\tau}}, \quad (3.62)$$

for the velocity and magnetic field

$$\mathbf{V}(t, \mathbf{r}) = Ax \mathbf{e}_y + iV_A \sum_{\mathbf{Q}} \mathbf{v}_{\mathbf{Q}}(\tau) e^{i\mathbf{Q} \cdot \mathbf{X}}, \quad (3.63)$$

$$\mathbf{B}(t, \mathbf{r}) = B_0 \boldsymbol{\alpha} + B_0 \sum_{\mathbf{Q}} \mathbf{b}_{\mathbf{Q}}(\tau) e^{i\mathbf{Q} \cdot \mathbf{X}}, \quad (3.64)$$

$$(3.65)$$

$$\sum_{\mathbf{Q}} = \int d\left(\frac{\mathbf{Q}}{2\pi}\right) = \int \frac{d^3 Q}{(2\pi)^3}, \quad \boldsymbol{\alpha} = (0, \alpha_y = \sin \theta, \alpha_z = \cos \theta), \quad (3.66)$$

$$V_A = \frac{B_0}{\sqrt{\mu_0 \rho}}, \quad \Lambda \equiv \frac{V_A}{A}, \quad \tau \equiv At. \quad (3.67)$$

$$\nu'_m = \nu_m / \Lambda V_A, \quad \nu'_k = \nu_k / \Lambda V_A, \quad \nu_m = \varepsilon_0 c^2 \varrho_{\Omega}, \quad \nu_k = \frac{\eta}{\rho}. \quad (3.68)$$

Here we use the self-explaining notations for the Alfvén velocity V_A , shear rate with dimension of frequency A , angular velocity of the fluid $\boldsymbol{\Omega} = A\omega_c \mathbf{e}_z$, kinematic viscosity ν_k and magnetic diffusivity ν_m . The θ is the angle between the normal of the shear flow plane (x, y) and magnetic field $\mathbf{B}_0 = B_0 \boldsymbol{\alpha}$. This problem is inspired from the physics of accretion disks where z is the rotation axis. Our goal is to calculate the effective viscosity

$$\nu_{\text{eff}}(\tau) = \nu_k + \nu_{\text{wave}}, \quad \nu_{\text{wave}} \equiv \nu_k \sum_{\mathbf{Q}} Q^2 \mathbf{v}_{\mathbf{Q}}^2(\tau) + \nu_m \sum_{\mathbf{Q}} Q^2 \mathbf{b}_{\mathbf{Q}}^2(\tau), \quad (3.69)$$

in the spirit of Boussinesq [Boussinesq (1877)] and Prandtl [Prandtl (1925)], when for static wave-turbulence we need of the time averaged spectral densities $\langle \mathbf{v}_Q^2 \rangle$ and $\langle \mathbf{b}_Q^2 \rangle$.

As the general problem for “eddy” viscosity η_{eff} is frustratingly complex we can start with the simplest possible 2D case of “pure shear” Fig. 9 [[Balbus and Hawley (1998)]], $\Omega = 0$, toroidal magnetic field $\theta = \pi/2$ and evanescent dissipation $\nu'_{tot} = \nu'_k + \nu'_m \ll 1$. For ideal fluid with $\nu'_{tot} = 0$ the linearized equations have exact solution in the framework of Heun functions [Dimitrov et al. (2011)], because the MHD problem is reduced to an effective quantum mechanical problem.

We will use this analytical result for the static case $\mathbf{v}_Q(\tau) = \mathbf{v}_Q$ and $\mathbf{b}_Q(\tau) = \mathbf{b}_Q$. For the 2D case of $Q_z = 0$, $v_z = 0 = b_z$ using the substitution $b_x = \psi(\bar{\tau})/\sqrt{1 + \bar{\tau}^2}$ we have Schrödinger type equation

$$d_{\bar{\tau}}^2 \psi + \left[Q_y^2 - \frac{1}{(1 + \bar{\tau}^2)^2} \right] \psi = 0, \quad \bar{\tau} \equiv -\frac{Q_x}{Q_y}, \quad (3.70)$$

whose solution represent the 2D MHD variables

$$b_y(\bar{\tau}) = \bar{\tau} b_x(\bar{\tau}), \quad v_y(\bar{\tau}) = \bar{\tau} v_x(\bar{\tau}), \quad v_x = d_{\bar{\tau}} b_x / Q_y \quad (3.71)$$

for convenience we consider $\bar{\tau}$ and Q_y as independent variables in the 2D wave-vector space.

3.3 Interplay between dissipative and nonlinear terms

Searching a static solution of general MHD equations Eq. (3.57) an Eq. (3.59) we have express v_y from the y -component of Eq. (3.59), and substitution in the y -component of Eq. (3.57) gives

$$d_{\bar{\tau}}^2 b_y + \nu'_{tot} Q^2 d_{\bar{\tau}} b_y + [Q_\alpha^2 + 2\nu'_m \tau Q_y^2 + \nu'_m \nu'_k Q^4] b_y = d_{\bar{\tau}} N_b^y - Q_\alpha N_v^y - 2Q_\alpha n_y^2 v_x + \nu'_k Q N_b^y + (\nu'_k - \nu'_m) Q^2 b_x \quad (3.72)$$

Now let us analyze all irrelevant for $|\bar{\tau}| \gg 1$ terms:

- 1) the linear term $v_x/(1 + \bar{\tau}^2)$ is responsible for the amplification of SMW at $\bar{\tau} \sim 1$ but is negligible for $|\bar{\tau}| \gg 1$,
- 2) $\nu'_m \nu'_k$ is negligible for evanescent dissipation $\nu'_m \nu'_k \ll 1$,
- 3) $\nu'_m \tau$ will be never essential because for large enough $|\bar{\tau}| > 1/\nu'_m$ the wave amplitude will be exponentially small.

In such a way for $|\bar{\tau}| \gg 1$ we derive an effective oscillator equation with damping $\propto \nu'_{tot}$ and external force created by the nonlinear terms

$$[d_{\bar{\tau}}^2 + \gamma(\bar{\tau}) d_{\bar{\tau}} + Q_y^2] b_y = F_{ext}, \quad \gamma(\bar{\tau}) = \nu'_{tot} Q^2, \quad F_{ext} = d_{\bar{\tau}} N_b^y - Q_\alpha N_v^y. \quad (3.73)$$

For small friction $\nu'_{tot} \ll 1$ the left hand linear operator has the WKB Green function

$$[d_{\bar{\tau}}^2 + \gamma(\bar{\tau}) d_{\bar{\tau}} + Q_y^2] G(\bar{\tau} - \bar{\tau}_0) = \delta(\bar{\tau} - \bar{\tau}_0) \quad (3.74)$$

$$G(\bar{\tau} - \bar{\tau}_0) \approx \frac{1}{Q_y} \sin[Q_y(\bar{\tau} - \bar{\tau}_0)] \exp\left(-\frac{1}{2} \int_{\bar{\tau}_0}^{\bar{\tau}} d\tau \gamma(\tau)\right) \theta(\bar{\tau} - \bar{\tau}_0), \quad (3.75)$$

where

$$\frac{1}{2} \int_{\bar{\tau}_0}^{\bar{\tau}} \gamma(\tau) d\tau \approx \frac{\nu'_{\text{tot}}}{6} Q_y^2 (\bar{\tau}^3 - \bar{\tau}_0^3). \quad (3.76)$$

The solution of the oscillator equation

$$b_y(\bar{\tau}, Q_y) = \int_{-\infty}^{\bar{\tau}} G(\bar{\tau}, \bar{\tau}_0) F_{\text{ext}}(\bar{\tau}_0) d\bar{\tau}_0 \quad (3.77)$$

is the basis of the self-consistent theory of wave turbulence described in the next section. If we take into account a small dissipation in the MHD equations, we have large $\bar{\tau} \gg 1$ asymptotics for 2D case $Q_z = 0$, and evanescent friction $\nu'_{\text{tot}} \ll 1$

$$\psi \approx D_f e^{-\nu'_{\text{tot}} Q_y^2 \bar{\tau}^3 / 6} \cos(|Q_y| \bar{\tau} + \phi_f), \quad \bar{\tau} > 0 \quad (3.78)$$

$$b_y \approx \frac{\bar{\tau} \psi}{\sqrt{1 + \bar{\tau}^2}} \approx \psi \approx D_f e^{-\nu'_{\text{tot}} Q_y^2 \bar{\tau}^3 / 6} \cos(|Q_y| \bar{\tau} + \phi_f), \quad (3.79)$$

$$v_y \approx \bar{\tau} \frac{(1 + \bar{\tau}^2) d\bar{\tau} \psi - \bar{\tau} \psi}{Q_y (1 + \bar{\tau}^2)^{3/2}} \approx -D_f e^{-\nu'_{\text{tot}} Q_y^2 \bar{\tau}^3 / 6} \sin(|Q_y| \bar{\tau} + \phi_f). \quad (3.80)$$

In other words, after the shear created amplification the nonlinear terms are negligible and alfvénons decay as free particles. In this WKB asymptotic for the wave energy density in wave-vector space for $\bar{\tau} \gg 1$ we have

$$\mathbf{b}_Q^2 + \mathbf{v}_Q^2 \approx D_f^2 \exp \left(- \int_0^{\bar{\tau}} \gamma(\tau) d\tau \right) \approx D_f^2 e^{-\nu'_{\text{tot}} Q_y^2 \bar{\tau}^3 / 3}, \quad (3.81)$$

and supposing strong amplification $\mathcal{A} \gg 1$ we have

$$\nu'_{\text{tot}} \int_{-\infty}^{\infty} Q^2 (\mathbf{b}_Q^2 + \mathbf{v}_Q^2) dQ_x \approx |Q_y| (\mathbf{b}_Q^2 + \mathbf{v}_Q^2)|_f = |Q_y| D_f^2, \quad (3.82)$$

$$(\mathbf{b}_Q^2 + \mathbf{v}_Q^2)|_f = D_f^2 \quad (3.83)$$

Density of wave energy in Q -space immediately after amplification is determined by amplitude of damped magnetosonic wave. Initial energy of magnetosonic wave gradually dissipation in head due to viscous friction. Thus according Eq. (3.82) dissipation power for all Q_x can be presented as energy of amplified wave for $Q_x = 0$. If we use virial theorem whereby in magnetosonic waves we have continuous transformation of magnetic energy into mechanical $\langle \mathbf{v}_Q^2 \rangle \approx \langle \mathbf{b}_Q^2 \rangle$ then for large Prandtl numbers $\nu_{\text{m}} \ll \nu_k \approx \nu_{\text{tot}}$ equation Eq. (3.93) can be rewrite as

$$\nu_{\text{eff}} = \frac{\nu_{\text{eff}}}{\nu_k} = 1 + \nu'_{\text{wave}} = 1 + \frac{1}{2} \int_Q Q^2 (\mathbf{v}_Q^2 + \mathbf{b}_Q^2) \frac{dQ_x dQ_y dQ_z}{(2\pi)^3}, \quad (3.84)$$

In this approximation of strongly amplified waves with small decay rate we have

$$\nu'_{\text{wave}} \approx \frac{1}{2} \frac{1}{(2\pi)^3} \int |\mathbf{U}_{\text{shear}}(\mathbf{Q})| (\mathbf{b}_Q^2 + \mathbf{v}_Q^2)|_f dQ_y dQ_z \quad (3.85)$$

or in 2D case taking into account the symmetry

$$\nu'_{\text{wave}} \approx \frac{1}{(2\pi)^2} \int_0^\infty |\mathbf{U}_{\text{shear}}(\mathbf{Q})| (\mathbf{b}_{\mathbf{Q}}^2 + \mathbf{v}_{\mathbf{Q}}^2) \big|_{\text{f}} dQ_y \approx \frac{1}{4\pi^2} \int_0^\infty Q_y D_{\text{f}}^2(Q_y) dQ_y. \quad (3.86)$$

In other words the total heating power can be evaluated as the flux of the amplified waves after the amplification plane $Q_x = 0$. Every wave finally gives his energy to the fluid and effective viscosity can be evaluated as surface integral from the energy flux

$$\mathbf{S}(\mathbf{Q}) \equiv \mathbf{U}_{\text{shear}}(\mathbf{Q}) (\mathbf{b}_{\mathbf{Q}}^2 + \mathbf{v}_{\mathbf{Q}}^2) / (2\pi)^3, \quad \nu'_{\text{wave}} = \frac{1}{2} \oint \mathbf{S} \cdot d\mathbf{f}, \quad (3.87)$$

where $d\mathbf{f} = dQ_y dQ_z$ is the elementary area in momentum space after the wave amplification when waves become independent. In the next section we will derive a self-consistent chain of the equation of the amplitudes of the amplified waves $D_{\text{f}}(Q_y)$ and finally we will express the effective viscosity by the solution using Eq. (3.97).

Dissipative processes in MHD

$$\tilde{Q}_{\text{tot}} = \tilde{Q}_{\eta}^{\text{wave}} + \tilde{Q}_{\text{Ohm}}^{\text{wave}} + \tilde{Q}_{\eta}^{\text{shear}} = \eta_{\text{eff}} A^2 \quad (3.88)$$

$$\tilde{Q}_{\eta}^{\text{shear}} = \frac{\eta}{2} \left\langle (\partial_k V_i^{\text{shear}} + \partial_i V_k^{\text{shear}})^2 \right\rangle = \eta A^2 = \rho V_A^2 A \nu'_k \quad (3.89)$$

$$\tilde{Q}_{\eta}^{\text{wave}} = \frac{\eta}{2} \left\langle (\partial_k V_i^{\text{wavw}} + \partial_i V_k^{\text{wave}})^2 \right\rangle = \rho A V_A^2 \nu'_k \sum_Q Q^2 \mathbf{v}_{\mathbf{Q}}^2 \quad (3.90)$$

$$\tilde{Q}_{\text{Ohm}}^{\text{wave}} = \langle \mathbf{j} \cdot \mathbf{E} \rangle = \frac{1}{\mu_0 \sigma_{\text{Ohm}}} (\text{rot } \mathbf{B}^{\text{wave}})^2 = \rho A V_A^2 \nu'_m \sum_Q Q^2 \mathbf{b}_{\mathbf{Q}}^2 \quad (3.91)$$

$$\tilde{Q} = (\rho \nu_{\text{eff}}) A^2 = \eta_{\text{eff}} A^2 = \sigma_{xy} A \quad (3.92)$$

$$\nu_{\text{eff}} = \nu_k + \nu_k \sum_Q Q^2 \mathbf{v}_{\mathbf{Q}}^2 + \nu_m \sum_Q Q^2 \mathbf{b}_{\mathbf{Q}}^2 \quad (3.93)$$

$$Z = \frac{\eta_{\text{eff}}}{\eta} = \frac{\sigma_{xy}}{\eta A} = \frac{\tilde{Q}_{\text{tot}}}{\eta A^2} = 1 + \int_Q Q^2 \left(\mathbf{v}_{\mathbf{Q}}^2 + \frac{\mathbf{b}_{\mathbf{Q}}^2}{P_m} \right) \frac{dQ_x dQ_y dQ_z}{(2\pi)^3}, \quad (3.94)$$

$$\frac{1}{P_m} = \frac{\epsilon_0 c^2 \varrho}{\eta / \rho} \quad (3.95)$$

$$P_m \propto T^4, \quad \frac{1}{P_m} \ll 1, \quad \nu'_{\text{tot}} = \nu'_k + \nu'_m \approx \nu'_k$$

$$\nu_{\text{eff}} \approx \nu_k + \nu_{\text{tot}} \sum_Q Q^2 \mathbf{v}_{\mathbf{Q}}^2 = \nu_k + \nu_{\text{wave}} \quad (3.96)$$

Lets introduce density of wave energy in Q -space $w_{\mathbf{Q}} = \frac{1}{2}(\mathbf{v}_{\mathbf{Q}}^2 + \mathbf{b}_{\mathbf{Q}}^2)$, drift velocity of magnetosonic waves in Q -space $\mathbf{U}_{\text{shear}}(\mathbf{Q}) = -Q_y \mathbf{e}_x$ cf. 146 Sozopol !!!!!, and energy flux in Q -space $\mathbf{S}(\mathbf{Q}) = \frac{1}{2} \mathbf{U}_{\text{shear}}(\mathbf{Q}) w_{\mathbf{Q}} / (2\pi)^3$. Using Eq. (3.82) and virial theorem $\langle \mathbf{v}_{\mathbf{Q}}^2 \rangle \approx \langle \mathbf{b}_{\mathbf{Q}}^2 \rangle$ the equation for the viscosity created by the waves Eq. (3.96) is

$$\nu'_{\text{wave}} = \frac{1}{2} \int_Q Q^2 (\mathbf{v}_{\mathbf{Q}}^2 + \mathbf{b}_{\mathbf{Q}}^2) \frac{dQ_x dQ_y dQ_z}{(2\pi)^3} = \oint \mathbf{S} \cdot d\mathbf{f} = \frac{1}{(2\pi)^3} \int_0^\infty |\mathbf{U}_{\text{shear}}(\mathbf{Q})| (\mathbf{b}_{\mathbf{Q}}^2 + \mathbf{v}_{\mathbf{Q}}^2) dQ_y dQ_z, \quad (3.97)$$

where $d\mathbf{f} = -\text{sgn}(Q_y) dQ_y dQ_z \mathbf{e}_x$ is the elementary area in Q -space. This formula have simple physical point - dissipated power of waves is equal to the power “emitted” waves near plane of amplification. In 2D case we skip integration on $dQ_z/2\pi$ and formal substitute $Q_z = 0$. Thus for renorm factor of viscosity Z Eq. (3.94) formula Eq. (3.86) and Eq. (3.83) gives

$$Z = 1 + \frac{1}{(2\pi)^2 \nu'_k} \int_0^\infty Q_y D_f^2 dQ_y = 1 + \frac{R^2}{(2\pi)^2 \nu'_k} I_D = 1 + \frac{\Gamma^2(\frac{4}{3})}{128\pi^4 \nu'_k} \left(\frac{6}{\nu'_{\text{tot}}} \right)^{2/3} I_D, \\ I_D = \int_0^\infty Q_y \tilde{D}_f^2 dQ_y, \quad \tilde{D} = RD.$$

here we use $D_f^2(-Q_y) = D_f^2(Q_y)$.

For hot plasmas where $\nu'_{\text{tot}} \approx \nu'_k$

$$Z \equiv \frac{\eta_{\text{eff}}}{\eta_{\text{kin}}} \approx \frac{\Gamma^2(\frac{4}{3}) 6^{2/3} I_D}{128\pi^4} \frac{1}{\nu_k^{5/3}} = \frac{\Gamma^2(\frac{4}{3}) 6^{2/3} I_D}{128\pi^4} \left(\frac{\rho V_A^2}{A\eta} \right)^{5/3} \gg 1 \quad (3.98)$$

$$\sigma_{xy} = \frac{\Gamma(\frac{4}{3}) 2 6^{2/3} I_D}{128\pi^4} \frac{(\rho V_A^2)^{5/3}}{(A\eta)^{2/3}} \ll P. \quad (3.99)$$

In the next section we will calculate renorm factor of viscosity in approximation of evanescent viscosity $\nu'_k \ll 1$ for who $Z \gg 1$. Aim of the current work in with the nonlinear terms are introduced in MHD equations is to investigate in details simplest solvable case of self-sustain wave turbulence, when effective viscosity dramatically increases. In this simplest 2D case we will consider solutions without rotating $\omega_c = 0$. As we already mentioned the next section will have technically character.

$$Z = 1 + \frac{C_z}{\nu_{\text{tot}}^{5/3}} \approx C_z \left(\frac{V_A^2}{\nu_{\text{tot}} A} \right)^{5/3} \gg 1, \quad \eta_{\text{eff}} = Z(\eta = \rho_0 \nu_k) = C_z \rho_0 \nu_k \left(\frac{V_A^2}{\nu_k A} \right)^{5/3} \gg 1 \\ \sigma_{xy} = \eta_{\text{eff}} A = Z \eta A = C_z \rho_0 (\nu_k A) \left(\frac{V_A^2}{\nu_k A} \right)^{5/3} = C_z \rho_0 \frac{V_A^{10/3}}{\nu_k^{2/3} A^{2/3}} \\ \tilde{Q} = A \sigma_{xy} = C_z \rho_0 \frac{V_A^{10/3} A^{1/3}}{\nu_k^{2/3}} = C_z \frac{B_0^{10/3} A^{1/3}}{\nu_k^{2/3} \rho_0^{2/3} \mu_0^{5/3}}$$

Cold Plasma $\eta \approx 0$

$$\eta_{\text{eff}} = \frac{\Gamma(\frac{4}{3})^2 6^{2/3} I_D}{128\pi^4} \rho \frac{V_A^2}{A} \left(\frac{6V_A^2}{Ac^2\epsilon_0\rho\Omega} \right)^{2/3} \quad (3.100)$$

$$\sigma_{xy} = \eta_{\text{eff}} A = \frac{\Gamma(\frac{4}{3})^2 6^{2/3} I_D}{128\pi^4} \rho V_A^{10/3} \left(\frac{6}{Ac^2\epsilon_0\rho\Omega} \right)^{2/3} \ll P \quad (3.101)$$

$$\tilde{Q}_{\text{tot}} = \frac{\Gamma(\frac{4}{3})^2 6^{2/3} I_D}{128\pi^4} \rho V_A^{10/3} A^{1/3} \left(\frac{6}{c^2\epsilon_0\rho\Omega} \right)^{2/3} \quad (3.102)$$

Anomalous transport in accretion disks is created by self-sustained beams of SMW propagating in radial direction; figuratively lasing of alfvénons create compact astrophysical objects.

3.4 Dimension analysis of MRI

$$\omega_c = -\frac{2}{3}, \quad \nu_{\text{tot}} \rightarrow 0, \quad \partial_\tau w_{\mathbf{Q}} = 0, \quad w_{\mathbf{Q}} \sim 1, \quad |Q| \sim 1$$

$$\oint \mathbf{S} \cdot d\mathbf{f} \sim 1, \quad \tilde{Q} = p_B A = \sigma A = \eta_{\text{eff}} A^2, \quad V_A \ll c_s, \quad p_B \ll p = nT$$

$$p_B/p = 1/\beta \ll 1 \quad \text{high beta-plasmas}$$

$$\sigma_{xy} \sim p_B, \quad \sigma_{xy} = C_\omega p_B = C_\omega \frac{p}{\beta} = \alpha_{s-s} p \quad (3.103)$$

Conclusion: Dimension analysis shows that Shakura-Sunaev parameter is simple reciprocal of plasma beta.

$$\alpha_{s-s} = C_w/\beta, \quad \mathcal{R}e = \frac{V_A(\Lambda = V_A/A)}{\nu}, \quad \eta_{\text{eff}} = p_B/A$$

$$Z = \frac{\eta_{\text{eff}}}{\eta} = \frac{B_0^2}{\mu_0 \eta A} = \frac{B_0^2}{\mu_0 \rho \nu A} = \frac{V_A}{\nu A} = \frac{C_z}{\nu_{\text{tot}}^{\kappa(\omega)}}$$

$$\kappa(\omega_c = 0) = \frac{5}{3}, \quad \kappa(\omega_c = -\frac{2}{3}) = 1$$

3.5 Self-Sustained Chain of Equation

For the general 2D case the nonlinear terms in Eq. (3.72) read:

$$\mathbf{N}_v = \Pi^{\perp \mathbf{Q}} \cdot \sum_{\mathbf{Q}'} \left[\begin{pmatrix} v_{\mathbf{Q}'}^x v_{\mathbf{Q}-\mathbf{Q}'}^x & v_{\mathbf{Q}'}^x v_{\mathbf{Q}-\mathbf{Q}'}^y \\ v_{\mathbf{Q}'}^y v_{\mathbf{Q}-\mathbf{Q}'}^x & v_{\mathbf{Q}'}^y v_{\mathbf{Q}-\mathbf{Q}'}^y \end{pmatrix} + \begin{pmatrix} b_{\mathbf{Q}'}^x b_{\mathbf{Q}-\mathbf{Q}'}^x & b_{\mathbf{Q}'}^x b_{\mathbf{Q}-\mathbf{Q}'}^y \\ b_{\mathbf{Q}'}^y b_{\mathbf{Q}-\mathbf{Q}'}^x & b_{\mathbf{Q}'}^y b_{\mathbf{Q}-\mathbf{Q}'}^y \end{pmatrix} \right] \cdot \begin{pmatrix} Q_x \\ Q_y \end{pmatrix} \quad (3.104)$$

$$\mathbf{N}_b = \Pi^{\perp \mathbf{Q}} \cdot \sum_{\mathbf{Q}'} \left[\begin{pmatrix} b_{\mathbf{Q}'}^x v_{\mathbf{Q}-\mathbf{Q}'}^x & b_{\mathbf{Q}'}^x v_{\mathbf{Q}-\mathbf{Q}'}^y \\ b_{\mathbf{Q}'}^y v_{\mathbf{Q}-\mathbf{Q}'}^x & b_{\mathbf{Q}'}^y v_{\mathbf{Q}-\mathbf{Q}'}^y \end{pmatrix} - \begin{pmatrix} v_{\mathbf{Q}'}^x b_{\mathbf{Q}-\mathbf{Q}'}^x & v_{\mathbf{Q}'}^x b_{\mathbf{Q}-\mathbf{Q}'}^y \\ v_{\mathbf{Q}'}^y b_{\mathbf{Q}-\mathbf{Q}'}^x & v_{\mathbf{Q}'}^y b_{\mathbf{Q}-\mathbf{Q}'}^y \end{pmatrix} \right] \cdot \begin{pmatrix} Q_x \\ Q_y \end{pmatrix} \quad (3.105)$$

where $\Pi^{\perp \mathbf{Q}}$ is the projection operator and have

$$\Pi^{\perp \mathbf{Q}} = n_{yy} \begin{pmatrix} 1 & \tau \\ \tau & \tau^2 \end{pmatrix} \quad (3.106)$$

Taking into account the asymptotic

$$\lim_{\tau \rightarrow \infty} \mathbf{N}_b = \begin{pmatrix} 0 \\ \sum_{\mathbf{Q}'} (b_{\mathbf{Q}'}^y v_{\mathbf{Q}-\mathbf{Q}'}^y - v_{\mathbf{Q}'}^y b_{\mathbf{Q}-\mathbf{Q}'}^y) Q_y \end{pmatrix} \quad (3.107)$$

$$\lim_{\tau \rightarrow \infty} \mathbf{N}_v = \begin{pmatrix} 0 \\ \sum_{\mathbf{Q}'} (v_{\mathbf{Q}'}^y v_{\mathbf{Q}-\mathbf{Q}'}^y + b_{\mathbf{Q}'}^y b_{\mathbf{Q}-\mathbf{Q}'}^y) Q_y \end{pmatrix} \quad (3.108)$$

To calculate the time derive of y component of nonlinear term \mathbf{N}_b we use relation

$$\begin{aligned} d_{\bar{\tau}} N_b^y &= d_{\bar{\tau}} \left[\sum_{\mathbf{Q}'} (b_{\mathbf{Q}'}^y v_{\mathbf{Q}-\mathbf{Q}'}^y - v_{\mathbf{Q}'}^y b_{\mathbf{Q}-\mathbf{Q}'}^y) Q_y \right] = [(b_{\mathbf{Q}'}^y b_{\mathbf{Q}-\mathbf{Q}'}^y + v_{\mathbf{Q}'}^y v_{\mathbf{Q}-\mathbf{Q}'}^y) (Q_y - 2Q_y')] Q_y, \\ d_{\bar{\tau}} v_{\mathbf{Q}-\mathbf{Q}'}^y &= (Q_y - Q_y') b_{\mathbf{Q}-\mathbf{Q}'}^y, \quad d_{\bar{\tau}} b_{\mathbf{Q}-\mathbf{Q}'}^y = -(Q_y - Q_y') v_{\mathbf{Q}-\mathbf{Q}'}^y \end{aligned}$$

The resulting external force acting on the out oscillator we have

$$\begin{aligned} F_{\text{ext}} &= d_{\bar{\tau}} N_b^y - Q_y N_v^y + N_b^y Q^2 \nu_k' + N_b^x \\ &= \sum_{\mathbf{Q}'} [(b_{\mathbf{Q}'}^y b_{\mathbf{Q}-\mathbf{Q}'}^y + v_{\mathbf{Q}'}^y v_{\mathbf{Q}-\mathbf{Q}'}^y) (Q_y^2 - 2Q_y' Q_y) - Q_y^2 (v_{\mathbf{Q}'}^y v_{\mathbf{Q}-\mathbf{Q}'}^y + b_{\mathbf{Q}'}^y b_{\mathbf{Q}-\mathbf{Q}'}^y) + Q_y^3 \bar{\tau}^2 \nu_k' (b_{\mathbf{Q}'}^y v_{\mathbf{Q}-\mathbf{Q}'}^y - v_{\mathbf{Q}'}^y b_{\mathbf{Q}-\mathbf{Q}'}^y)] \\ &= \sum_{\mathbf{Q}'} [Q_y^3 \bar{\tau}^2 \nu_k' (b_{\mathbf{Q}'}^y v_{\mathbf{Q}-\mathbf{Q}'}^y - v_{\mathbf{Q}'}^y b_{\mathbf{Q}-\mathbf{Q}'}^y) - 2Q_y' Q_y (b_{\mathbf{Q}'}^y b_{\mathbf{Q}-\mathbf{Q}'}^y + v_{\mathbf{Q}'}^y v_{\mathbf{Q}-\mathbf{Q}'}^y)] . \end{aligned}$$

$$\lim_{\nu_k' \rightarrow 0} \left(\frac{\int_0^\infty \nu_k' \bar{\tau}^2 e^{-\nu_{\text{tot}}' Q_y^2 \bar{\tau}^2 / 6} d\bar{\tau}}{\int_0^\infty e^{-\nu_{\text{tot}}' Q_y^2 \bar{\tau}^2 / 6} d\bar{\tau}} = \frac{6^{2/3}}{3\Gamma(\frac{4}{3})} \frac{\nu_k'^{1/3}}{Q_y^{4/3} (1 + \frac{1}{P_m})^{2/3}} \right) = 0 \quad (3.110)$$

Finally the result in limit of hot plasma is

$$F_{\text{ext}} \approx -2Q_y \sum_{\mathbf{Q}'} Q_y' (b_{\mathbf{Q}'}^y b_{\mathbf{Q}-\mathbf{Q}'}^y + v_{\mathbf{Q}'}^y v_{\mathbf{Q}-\mathbf{Q}'}^y) \quad (3.111)$$

$$b_y(\bar{\tau}, Q_y) = \int_{-\infty}^{\bar{\tau}} \frac{1}{Q_y} \sin[Q_y(\bar{\tau} - \bar{\tau}_0)] \exp\left(-\frac{1}{2} \int_{\bar{\tau}_0}^{\bar{\tau}} d\tau \gamma(\tau)\right) \theta(\bar{\tau} - \bar{\tau}_0) \cdot 2Q_y \sum_{\mathbf{Q}'} Q'_y (b_{\mathbf{Q}'}^y b_{\mathbf{Q}-\mathbf{Q}'}^y + v_{\mathbf{Q}'}^y v_{\mathbf{Q}-\mathbf{Q}'}^y) d\bar{\tau}_0$$

Now we will represent b_y as a sum sine and cosine terms $b_y(\bar{\tau}, Q_y) = \sin(Q_y \bar{\tau})X(\bar{\tau}) - \cos(Q_y \bar{\tau})Y(\bar{\tau})$

$$\begin{aligned} b_y(\bar{\tau}, Q_y) &= \int_{-\infty}^{\infty} \frac{1}{Q_y} \sin[Q_y \bar{\tau} - Q_y \bar{\tau}_0] \exp\left[-\frac{\nu'_{tot}}{6}(\bar{\tau}^3 - \bar{\tau}_0^3)W_y^2\right] \theta(\bar{\tau} - \bar{\tau}_0) \mathbf{N}_{bb}^{yy}(Q_x, Q_y) d\bar{\tau}_0 \\ &= \sin(Q_y \bar{\tau}) \left\{ \int_{-\infty}^{\bar{\tau}} d\bar{\tau}_0 \frac{1}{Q_y} \cos(Q_y \bar{\tau}_0) \exp\left[-\frac{\nu'_{tot}}{6}(\bar{\tau}^3 - \bar{\tau}_0^3)\right] \mathbf{N}_{bb}^{yy}(\mathbf{Q}) \equiv Y(\bar{\tau}) \right\} \\ &\quad - \cos(Q_y \bar{\tau}) \left\{ \int_{-\infty}^{\bar{\tau}} d\bar{\tau}_0 \frac{1}{Q_y} \sin(Q_y \bar{\tau}_0) \exp\left[-\frac{\nu'_{tot}}{6}(\bar{\tau}^3 - \bar{\tau}_0^3)\right] \mathbf{N}_{bb}^{yy}(\mathbf{Q}) \equiv -X(\bar{\tau}) \right\} \\ &= \cos(Q_y \bar{\tau})X(\bar{\tau}) + \sin(Q_y \bar{\tau})Y(\bar{\tau}) \approx X_0 \cos(Q_y \bar{\tau}) + Y_0 \sin(Q_y \bar{\tau}) \end{aligned}$$

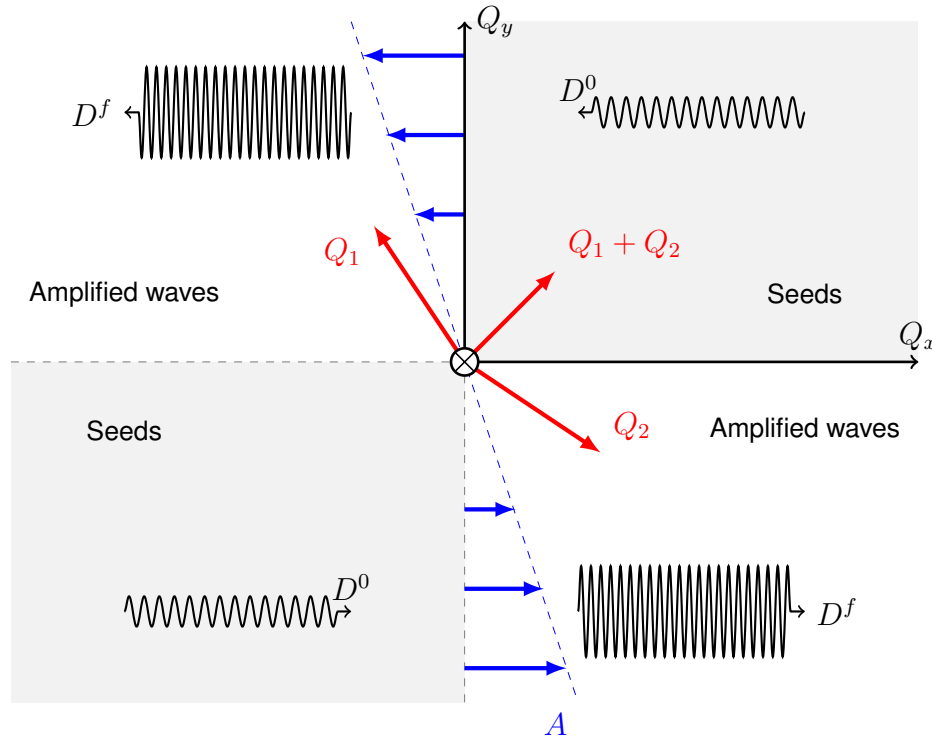
We can assume that the amplitude of the amplified wave as a function of $\bar{\tau}$ is approximately equal to amplitude in $\bar{\tau}_0 = 0$.

$$WKB \rightarrow X(\bar{\tau}) \approx X(0), \quad Y(\bar{\tau}) \approx Y(0); \quad \bar{\tau} - \bar{\tau}_0 = 0$$

$$X(0) = X_0 = - \int_{-\infty}^0 d\bar{\tau}_0 \frac{1}{Q_y} \sin(Q_y \bar{\tau}_0) \exp\left[-\frac{\nu'_{tot}}{6}(-\bar{\tau})^3 Q_y^2\right] \mathbf{N}_{bb}^{yy}(\mathbf{Q}) \quad (3.112)$$

$$Y_0 = Y(\bar{\tau} = 0) = \int_{-\infty}^0 d\bar{\tau}_0 \frac{1}{Q_y} \cos(Q_y \bar{\tau}_0) \exp\left[-\frac{\nu'_{tot}}{6}(-\bar{\tau})^3 Q_y^2\right] \mathbf{N}_{bb}^{yy}(\mathbf{Q}) \quad (3.113)$$

$$\begin{aligned} X(0) &= - \int_{-\infty}^0 \frac{dQ_x}{Q_y} \sin(Q_x) \exp\left[-\frac{\nu'_{tot}}{6}\bar{\tau}^3 Q_y^2\right] \cdot \sum_{\mathbf{Q}'} Q'_y (b_{\mathbf{Q}'}^y b_{\mathbf{Q}-\mathbf{Q}'}^y + v_{\mathbf{Q}'}^y v_{\mathbf{Q}-\mathbf{Q}'}^y) \\ Y(0) &= \int_{-\infty}^0 \frac{dQ_x}{Q_y} \cos(Q_x) \exp\left[-\frac{\nu'_{tot}}{6}\bar{\tau}^3 Q_y^2\right] \cdot \sum_{\mathbf{Q}'} Q'_y (b_{\mathbf{Q}'}^y b_{\mathbf{Q}-\mathbf{Q}'}^y + v_{\mathbf{Q}'}^y v_{\mathbf{Q}-\mathbf{Q}'}^y) \end{aligned} \quad (3.114)$$



In calculation of wave-wave interaction we will use the following scheme:

Amplified wave with wave-vector \mathbf{Q}_1 interact with another amplified wave with wave-vector \mathbf{Q}_2 and in the result we have wave with wave-vector $\mathbf{Q} = \mathbf{Q}_1 + \mathbf{Q}_2$.

Domain of amplification

$$Q_1^x < 0, \quad Q_2^x > 0, \quad Q_1^y > 0, \quad Q_2^y < 0$$

$$Q_1^x = \frac{Q_x}{2} + P_x, \quad Q_2^x = \frac{Q_x}{2} - P_x, \quad Q_1^y = \frac{Q_y}{2} + P_y, \quad Q_2^y = \frac{Q_y}{2} - P_y$$

$$\mathbf{P} \rightarrow \mathbf{K}, (P_x, P_y) = (-K_x, K_y)$$

$$Q_1^x = -\left(K_x - \frac{Q_x}{2}\right) < 0, \quad Q_2^x = +\left(K_x + \frac{Q_x}{2}\right) > 0, \quad (3.115)$$

$$Q_1^y = +\left(K_y + \frac{Q_y}{2}\right) > 0, \quad Q_2^y = -\left(K_y - \frac{Q_y}{2}\right) < 0 \quad (3.116)$$

$$K_x \in \left(\frac{Q_x}{2}, \infty\right), \quad K_y \in \left(\frac{Q_y}{2}, \infty\right) \quad (3.117)$$

After change of variables we have

$$I = \int_{Q_x/2}^{\infty} \frac{dK_x}{2\pi} \int_{Q_y/2}^{\infty} \frac{dK_y}{2\pi} Q_1^y (b_{Q_1}^y b_{Q_2}^y + v_{Q_1}^y v_{Q_2}^y). \quad (3.118)$$

Let us introduce some short notations

$$\begin{aligned} D^+ &\equiv D(Q_1^y) = D\left(K_y + \frac{Q_y}{2}\right), & D^- &\equiv D(Q_2^y) = D\left(\frac{Q_y}{2} - K_y\right), \\ \phi^+ &\equiv \phi(Q_1^y) = \phi\left(K_y + \frac{Q_y}{2}\right), & \phi^- &\equiv \phi(Q_2^y) = \phi\left(\frac{Q_y}{2} - K_y\right). \end{aligned} \quad (3.119)$$

Then in explicit form we have for $b_{Q_1}^y b_{Q_2}^y$

$$D^+ \cos [Q_1^y \bar{\tau}_1 + \phi^+] \exp\left[-\frac{\nu'_{tot}}{6} (Q_1^y)^2 \bar{\tau}_1^3\right] \cdot D^- \cos [Q_2^y \bar{\tau}_2 + \phi^-] \exp\left[-\frac{\nu'_{tot}}{6} (Q_2^y)^2 \bar{\tau}_2^3\right],$$

and respectively for $v_{Q_1}^y v_{Q_2}^y$

$$D^+ \sin [Q_1^y \bar{\tau}_1 + \phi^+] \exp\left[-\frac{\nu'_{tot}}{6} (Q_1^y)^2 \bar{\tau}_1^3\right] \cdot D^- \sin [Q_2^y \bar{\tau}_2 + \phi^-] \exp\left[-\frac{\nu'_{tot}}{6} (Q_2^y)^2 \bar{\tau}_2^3\right].$$

The argument of exponential function in both of relations above is strictly negative

$$\begin{aligned} (Q_1^y)^2 \bar{\tau}_1^3 &= -\frac{(Q_1^x)^3}{Q_1^y} = \left(\frac{(K_x - Q_x/2)^3}{K_y + Q_y/2}\right) > 0, \\ (Q_2^y)^2 \bar{\tau}_2^3 &= -\frac{(Q_2^x)^3}{Q_2^y} = \left(\frac{(K_x + Q_x/2)^3}{K_y - Q_y/2}\right) > 0. \end{aligned}$$

The resulting exponent function in expression $b_{Q_1}^y b_{Q_2}^y + v_{Q_1}^y v_{Q_2}^y$ is

$$\exp\left[-\frac{\nu'_{tot}}{6} (Q_1^y)^2 \bar{\tau}_1^3\right] \cdot \exp\left[-\frac{\nu'_{tot}}{6} (Q_2^y)^2 \bar{\tau}_2^3\right] = \exp\left\{-\frac{\nu'_{tot}}{6} \left[\frac{(K_x - Q_x/2)^3}{K_y + Q_y/2} + \frac{(K_x + Q_x/2)^3}{K_y - Q_y/2}\right]\right\}$$

To calculate trigonometrical part of Eq. (3.118) we use formula

$$\cos(\alpha) \cos(\beta) + \sin(\alpha) \sin(\beta) = \cos(\alpha - \beta),$$

and for our case obtain

$$\begin{aligned} &\cos(Q_1^y \bar{\tau}_1 + \phi^+) \cos(Q_2^y \bar{\tau}_2 + \phi^-) + \sin(Q_1^y \bar{\tau}_1 + \phi^+) \sin(Q_2^y \bar{\tau}_2 + \phi^-) = \cos[2K_x + \phi^+ - \phi^-] \\ &= \cos(2K_x) \{ \cos[\phi(Q_1^y)] \cos[\phi(Q_2^y)] + \sin[\phi(Q_1^y)] \sin[\phi(Q_2^y)] \} \\ &- \sin(2K_x) \{ \sin[\phi(Q_1^y)] \cos[\phi(Q_2^y)] - \cos[\phi(Q_1^y)] \sin[\phi(Q_2^y)] \} \end{aligned}$$

Finlay we can assemble all calculated terms for $Q_1^y (b_{Q_1}^y b_{Q_2}^y + v_{Q_1}^y v_{Q_2}^y)$

$$Q_1^y D^+ D^- \cos[2K_x + \phi^+ - \phi^-] \exp \left\{ -\frac{\nu'_{tot}}{6} \left[\frac{(K_x - Q_x/2)^3}{K_y + Q_y/2} + \frac{(K_x + Q_x/2)^3}{K_y - Q_y/2} \right] \right\}$$

Now our expression for I is

$$\begin{aligned} I = & \int_{Q_x/2}^{\infty} \exp \left\{ -\frac{\nu'_{tot}}{6} \left[\frac{(K_x - Q_x/2)^3}{K_y + Q_y/2} + \frac{(K_x + Q_x/2)^3}{K_y - Q_y/2} \right] \right\} \cos(2K_x) \frac{dK_x}{2\pi} \cdot \\ & \int_{Q_y/2}^{\infty} \frac{dK_y}{2\pi} Q_1^y D^+ D^- \{ \cos \phi^+ \cos \phi^- + \sin \phi^+ \sin \phi^- \} \\ & - \int_{Q_x/2}^{\infty} \exp \left\{ -\frac{\nu'_{tot}}{6} \left[\frac{(K_x - Q_x/2)^3}{K_y + Q_y/2} + \frac{(K_x + Q_x/2)^3}{K_y - Q_y/2} \right] \right\} \sin(2K_x) \frac{dK_x}{2\pi} \cdot \\ & \int_{Q_y/2}^{\infty} \frac{dK_y}{2\pi} Q_1^y D^+ D^- \{ \sin \phi^+ \cos \phi^- - \cos \phi^+ \sin \phi^- \} \end{aligned}$$

In order to extract the viscosity outside of the integral we introduce following variables

$$K_x \equiv \left(\frac{6}{\nu'_{tot}} \right)^{1/3} x, \quad Q_x \equiv \left(\frac{6}{\nu'_{tot}} \right)^{1/3} q_x, \quad \omega = 2 \left(\frac{6}{\nu'_{tot}} \right)^{1/3}.$$

The integral over dK can be represent as a sum of fast oscillations integral form 0 to ∞ and simple integral of trigonometrical function in finite interval. In the second integral exponential function doesn't affect the result because the viscosity ν'_{tot} tends to zero.

$$\begin{aligned} & \int_{Q_x/2}^{\infty} \exp \left\{ -\frac{\nu'_{tot}}{6} \left[\frac{(K_x - Q_x/2)^3}{K_y + Q_y/2} + \frac{(K_x + Q_x/2)^3}{K_y - Q_y/2} \right] \right\} \cos(2K_x) \frac{dK_x}{2\pi} = \quad (3.120) \\ & \frac{1}{2\pi} \left(\frac{6}{\nu'_{tot}} \right)^{1/3} \int_0^{\infty} \exp \left\{ -\left[\frac{(x - q_x/2)^3}{K_y + Q_y/2} + \frac{(x + q_x/2)^3}{K_y - Q_y/2} \right] \right\} \cos(2\omega x) dx - \\ & \int_0^{Q_x/2} \exp \left\{ -\frac{\nu'_{tot}}{6} \left[\frac{(K_x - Q_x/2)^3}{K_y + Q_y/2} + \frac{(K_x + Q_x/2)^3}{K_y - Q_y/2} \right] \right\} \cos(2K_x) \frac{dK_x}{2\pi} \end{aligned}$$

$$\begin{aligned} & \lim_{\nu'_{tot} \rightarrow \infty} \left\{ \frac{1}{2\pi} \left(\frac{6}{\nu'_{tot}} \right)^{1/3} \int_0^{\infty} \exp \left\{ -\left[\frac{(x - q_x/2)^3}{K_y + Q_y/2} + \frac{(x + q_x/2)^3}{K_y - Q_y/2} \right] \right\} \cos(\omega x) dx \right\} = \\ & \frac{1}{8\pi} \left(\frac{\nu'_{tot}}{6} \right)^{1/3} \frac{d}{dx} \left(\exp \left\{ -\left[\frac{(x - q_x/2)^3}{K_y + Q_y/2} + \frac{(x + q_x/2)^3}{K_y - Q_y/2} \right] \right\} \right) \Big|_0 = 0 \quad (3.121) \end{aligned}$$

$$\begin{aligned} & \lim_{\nu'_{tot} \rightarrow 0} \left\{ \int_0^{Q_x/2} \exp \left\{ -\frac{\nu'_{tot}}{6} \left[\frac{(K_x - Q_x/2)^3}{K_y + Q_y/2} + \frac{(K_x + Q_x/2)^3}{K_y - Q_y/2} \right] \right\} \cos(2K_x) \frac{dK_x}{2\pi} \right\} = \\ & \int_0^{Q_x/2} \cos(2K_x) \frac{dK_x}{2\pi} = \frac{1}{4\pi} \sin(Q_x) \quad (3.122) \end{aligned}$$

$$\lim_{\omega \rightarrow \infty} \left\{ \frac{1}{2\pi} \left(\frac{6}{\nu'_{\text{tot}}} \right)^{1/3} \int_0^\infty \exp \left\{ - \left[\frac{(x - q_x/2)^3}{K_y + Q_y/2} + \frac{(x + q_x/2)^3}{K_y - Q_y/2} \right] \right\} \sin(\omega x) dx \right\} =$$

$$\frac{1}{8\pi} \left(\exp \left\{ - \left[\frac{(-q_x/2)^3}{K_y + Q_y/2} + \frac{(+q_x/2)^3}{K_y - Q_y/2} \right] \right\} \right) = \frac{1}{8\pi} \exp \left\{ -q_x^3 \frac{Q_y}{2(4K_y^2 - Q_y^2)} \right\} \quad (3.123)$$

$$\lim_{\nu'_{\text{tot}} \rightarrow 0} \left\{ \int_0^{Q_x/2} \exp \left\{ - \frac{\nu'_{\text{tot}}}{6} \left[\frac{(K_x - Q_x/2)^3}{K_y + Q_y/2} + \frac{(K_x + Q_x/2)^3}{K_y - Q_y/2} \right] \right\} \sin(2K_x) \frac{dK_x}{2\pi} \right\} =$$

$$\int_0^{Q_x/2} \sin(2K_x) \frac{dK_x}{2\pi} = \frac{1}{4\pi} [\cos(Q_x) - 1] \quad (3.124)$$

Finlay we can assemble previously calculated results for I

$$I = -\frac{1}{4\pi} \sin(Q_x) \int_{Q_y/2}^\infty \frac{dK_y}{2\pi} Q_1^y D^+ D^- \{ \cos \phi^+ \cos \phi^- + \sin \phi^+ \sin \phi^- \} +$$

$$\frac{1}{4\pi} [\cos(Q_x) - 1] \int_{Q_y/2}^\infty \frac{dK_y}{2\pi} Q_1^y D^+ D^- \{ \sin \phi^+ \cos \phi^- - \cos \phi^+ \sin \phi^- \}$$

$$X = \frac{1}{4\pi Q_y} \left(\frac{6}{\nu'_{\text{tot}}} \right)^{1/3} \int_0^\infty \frac{1}{2} \exp \left\{ - \frac{q_x^3}{Q_y} \right\} dq_x \int_{Q_y/2}^\infty \frac{dK_y}{2\pi} Q_1^y D^+ D^- \{ \cos \phi^+ \cos \phi^- + \sin \phi^+ \sin \phi^- \}$$

$$Y = \frac{1}{4\pi Q_y} \left(\frac{6}{\nu'_{\text{tot}}} \right)^{1/3} \int_0^\infty \frac{1}{2} \exp \left\{ - \frac{q_x^3}{Q_y} \right\} dq_x \int_{Q_y/2}^\infty \frac{dK_y}{2\pi} Q_1^y D^+ D^- \{ \sin \phi^+ \cos \phi^- - \cos \phi^+ \sin \phi^- \}$$

After integration over dq_x we have

$$X = \frac{\Gamma(\frac{4}{3})}{16\pi Q_y^{2/3}} \left(\frac{6}{\nu'_{\text{tot}}} \right)^{1/3} \int_{Q_y/2}^\infty Q_1^y D^+ D^- \{ \cos \phi^+ \cos \phi^- + \sin \phi^+ \sin \phi^- \} dK_y$$

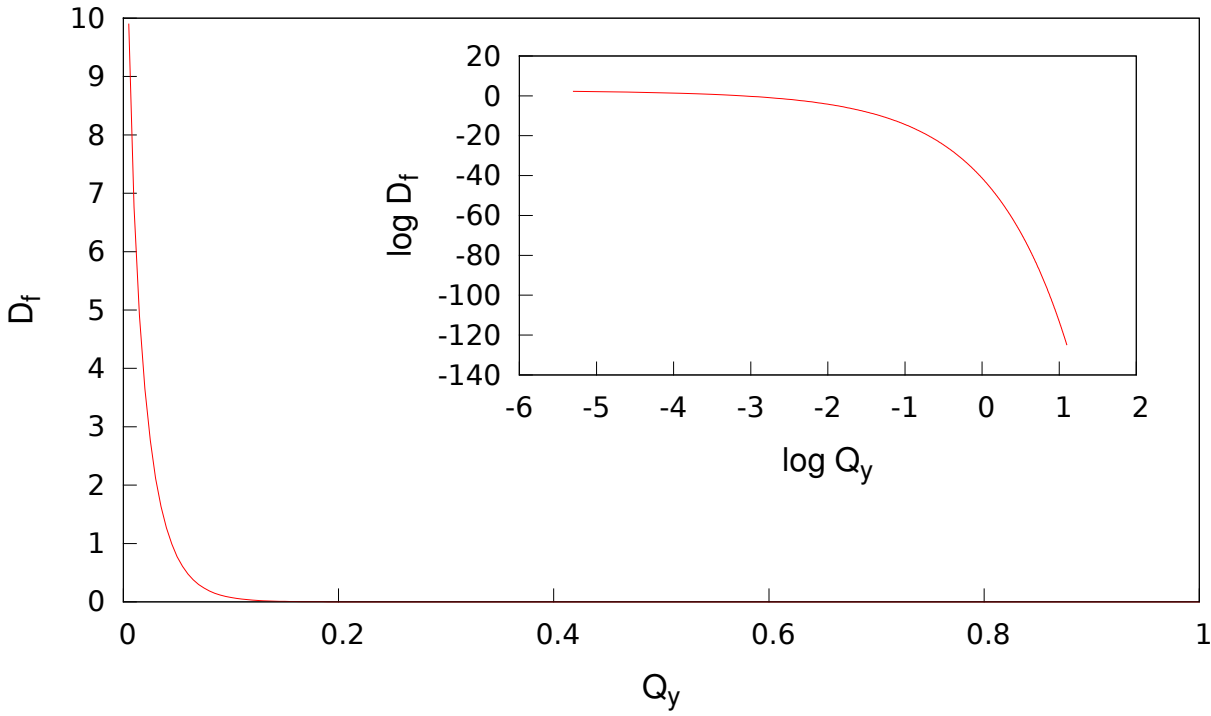
$$Y = \frac{\Gamma(\frac{4}{3})}{16\pi Q_y^{2/3}} \left(\frac{6}{\nu'_{\text{tot}}} \right)^{1/3} \int_{Q_y/2}^\infty Q_1^y D^+ D^- \{ \sin \phi^+ \cos \phi^- - \cos \phi^+ \sin \phi^- \} dK_y$$

$$R \equiv \frac{\Gamma(\frac{4}{3})}{16\pi} \left(\frac{6}{\nu'_{\text{tot}}} \right)^{1/3}, \quad \tilde{X} \equiv XR, \quad \tilde{Y} \equiv YR, \quad \tilde{D} \equiv DR.$$

Here we us exclude viscosity in our equation using scale transformation with R

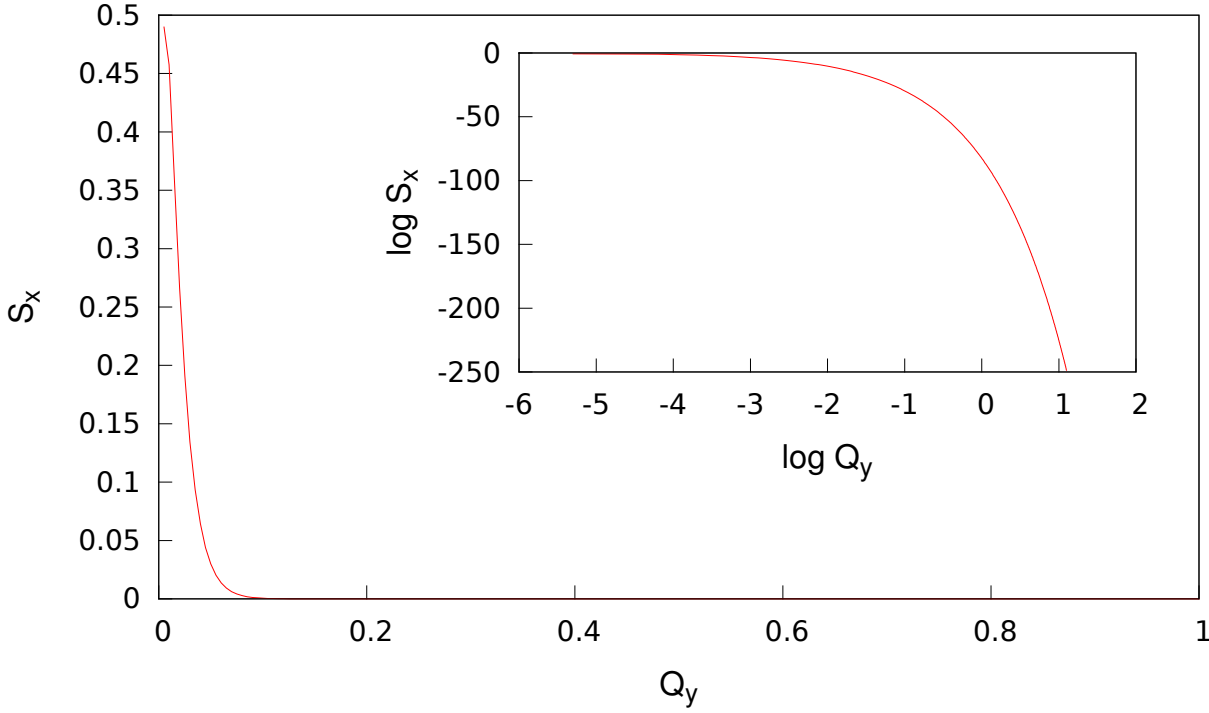
$$\tilde{X}_i(Q_y) = \int_{Q_y/2}^\infty \left(\frac{K_y}{Q_y^{2/3}} + \frac{Q_y^{1/3}}{2} \right) \tilde{D}_+ \tilde{D}_- [\cos \phi_+ \cos \phi_- + \sin \phi_+ \sin \phi_-] dK_y \quad (3.125)$$

$$\tilde{Y}_i(Q_y) = \int_{Q_y/2}^\infty \left(\frac{K_y}{Q_y^{2/3}} + \frac{Q_y^{1/3}}{2} \right) \tilde{D}_+ \tilde{D}_- [\sin \phi_+ \cos \phi_- - \cos \phi_+ \sin \phi_-] dK_y \quad (3.126)$$

Figure 3.1: $D_f(Q_y)$

$$\nu_{\text{wave}} \approx \frac{R}{4\pi^2} \int_0^\infty Q_y \tilde{D}_f^2(Q_y) dQ_y, \quad I_D = \int_0^\infty Q_y \tilde{D}_f^2(Q_y) dQ_y \quad (3.127)$$

We perform numerical analysis of the solution of the general MHD equations in the simplest case of 2D non-rotating plasma. The solution obtained in approximation of small viscosity or large Reynolds numbers \mathcal{R} show that self-sustained wave turbulence consist of two narrow beams with high wave density in Q -space. For large enough Reynolds numbers this beams or jets are converted in rays of magnetohydrodynamic waves in Q -space. We believe that this bright expressed phenomena qualitatively will be preserved in 3D case when rotation of plasma is taking into account. On this way solving of the MHD equations will reveal mechanism of anomalous transport in magnetized space plasmas. It is matter of numerical analysis to check, that giant effective viscosity and friction forces in accretion disks is determined by lasing of alfvénons in radial direction.

Figure 3.2: $S_x(Q_y)$

3.6 3D rotation case with Coriolis force

$$\begin{aligned}
 & d_{\bar{\tau}}^2 b_x + [\nu'_{\text{tot}} Q^2 - 2n_x n_y (1 + \omega_c)] d_{\bar{\tau}} b_x + [Q_\alpha^2 + 2\tau \nu'_m Q_y^2 (1 + \omega_c) + \nu'_k \nu'_m Q_y^4] b_x \\
 & = d_{\bar{\tau}} \mathbf{N}_b^x - Q_\alpha \mathbf{N}_v^x + 2\omega_c Q_\alpha v_y (n_x n_x + 1) - \mathbf{N}_b^x [2n_x n_y (1 + \omega_c) - \nu'_k Q^2] \\
 & d_{\bar{\tau}}^2 b_y + [\nu'_{\text{tot}} Q^2 + 2n_x n_y \omega_c] d_{\bar{\tau}} b_y + [Q_\alpha^2 + 2\tau \nu'_m Q_y^2 (1 - \omega_c) + \nu'_k \nu'_m Q_y^4] b_y \\
 & = \mathbf{N}_b^y - \mathbf{N}_v^y Q_\alpha + N_b^y (\nu'_k Q^2 + 2\omega_c n_x n_y) + \mathbf{N}_b^x - \nu'_m Q^2 b_x - 2Q_\alpha v_x [n_x n_y + \omega_c (n_y n_y - 1)]
 \end{aligned} \tag{3.128}$$

Obviously the approximate solution Eq. (2.103) can't be use for further analytical analysis. We can obtain asymptotic for the approximate solution in terms of spherical Bessel function of the second kind, hypergeometric function, but using just sine asymptotic with taking into account phase and amplitude is better way to solve the problem because we will have universal description and technique in terms of simple trigonometrical functions.

$$\lim_{\tau \rightarrow \infty} \mathbf{N}_b = \begin{pmatrix} 0 \\ \sum_{\mathbf{Q}'} (b_{\mathbf{Q}'}^y v_{\mathbf{Q}-\mathbf{Q}'}^y - v_{\mathbf{Q}'}^y b_{\mathbf{Q}-\mathbf{Q}'}^y) Q_y + (b_{\mathbf{Q}'}^y v_{\mathbf{Q}-\mathbf{Q}'}^z - v_{\mathbf{Q}'}^y b_{\mathbf{Q}-\mathbf{Q}'}^z) Q_z \\ \sum_{\mathbf{Q}'} (b_{\mathbf{Q}'}^z v_{\mathbf{Q}-\mathbf{Q}'}^y - v_{\mathbf{Q}'}^z b_{\mathbf{Q}-\mathbf{Q}'}^y) Q_y + (b_{\mathbf{Q}'}^z v_{\mathbf{Q}-\mathbf{Q}'}^z - v_{\mathbf{Q}'}^z b_{\mathbf{Q}-\mathbf{Q}'}^z) Q_z \end{pmatrix} \quad (3.129)$$

$$\lim_{\tau \rightarrow \infty} \mathbf{N}_v = \begin{pmatrix} 0 \\ \sum_{\mathbf{Q}'} (v_{\mathbf{Q}'}^y v_{\mathbf{Q}-\mathbf{Q}'}^y + b_{\mathbf{Q}'}^y b_{\mathbf{Q}-\mathbf{Q}'}^y) Q_y + (v_{\mathbf{Q}'}^y v_{\mathbf{Q}-\mathbf{Q}'}^z + b_{\mathbf{Q}'}^y b_{\mathbf{Q}-\mathbf{Q}'}^z) Q_z \\ \sum_{\mathbf{Q}'} (v_{\mathbf{Q}'}^z v_{\mathbf{Q}-\mathbf{Q}'}^y + b_{\mathbf{Q}'}^z b_{\mathbf{Q}-\mathbf{Q}'}^y) Q_y + (v_{\mathbf{Q}'}^z v_{\mathbf{Q}-\mathbf{Q}'}^z + b_{\mathbf{Q}'}^z b_{\mathbf{Q}-\mathbf{Q}'}^z) Q_z \end{pmatrix} \quad (3.130)$$

To calculate the time derive of y component of nonlinear term \mathbf{N}_b we use relation

$$\begin{aligned} d_{\bar{\tau}} N_b^y &= d_{\bar{\tau}} \left[\sum_{\mathbf{Q}'} (b_{\mathbf{Q}'}^y v_{\mathbf{Q}-\mathbf{Q}'}^y - v_{\mathbf{Q}'}^y b_{\mathbf{Q}-\mathbf{Q}'}^y) Q_y + (b_{\mathbf{Q}'}^y v_{\mathbf{Q}-\mathbf{Q}'}^z - v_{\mathbf{Q}'}^y b_{\mathbf{Q}-\mathbf{Q}'}^z) Q_z \right] \\ &= \sum_{\mathbf{Q}'} [(b_{\mathbf{Q}'}^y b_{\mathbf{Q}-\mathbf{Q}'}^y + v_{\mathbf{Q}'}^y v_{\mathbf{Q}-\mathbf{Q}'}^y) Q_y + (b_{\mathbf{Q}'}^y b_{\mathbf{Q}-\mathbf{Q}'}^z + v_{\mathbf{Q}'}^y v_{\mathbf{Q}-\mathbf{Q}'}^z) Q_z] (Q_\alpha - 2Q'_\alpha), \\ d_{\bar{\tau}} v_{\mathbf{Q}-\mathbf{Q}'}^y &= (Q_\alpha - Q'_\alpha) b_{\mathbf{Q}-\mathbf{Q}'}^y, \quad d_{\bar{\tau}} b_{\mathbf{Q}-\mathbf{Q}'}^y = -(Q_\alpha - Q'_\alpha) v_{\mathbf{Q}-\mathbf{Q}'}^y \end{aligned}$$

The resulting external force acting on the out oscillator we have

$$\begin{aligned} F_{\text{ext}} &= d_{\bar{\tau}} N_b^y - Q_\alpha N_v^y \\ &= -2 \sum_{\mathbf{Q}'} Q'_\alpha [(b_{\mathbf{Q}'}^y b_{\mathbf{Q}-\mathbf{Q}'}^y + v_{\mathbf{Q}'}^y v_{\mathbf{Q}-\mathbf{Q}'}^y) Q_y + (b_{\mathbf{Q}'}^y b_{\mathbf{Q}-\mathbf{Q}'}^z + v_{\mathbf{Q}'}^y v_{\mathbf{Q}-\mathbf{Q}'}^z) Q_z] \end{aligned} \quad (3.131)$$

Here we can use incompressibility condition to express z component of magnetic field and velocity from x and y components. In order to simplify the consideration we can use phase portraits from previous chapter to motivate the approximation

$$F_{\text{ext}} \approx -4Q_y \sum_{\mathbf{Q}'} Q'_\alpha (b_{\mathbf{Q}'}^y b_{\mathbf{Q}-\mathbf{Q}'}^y + v_{\mathbf{Q}'}^y v_{\mathbf{Q}-\mathbf{Q}'}^y) \quad (3.132)$$

Thus we obtain important result: nonlinear terms for general 3D case with Coriolis force have in approximation the same form as nonlinear term for 2D pure shear case.

$$b_y(\bar{\tau}, Q_y, Q_z) = \int_{-\infty}^{\bar{\tau}} \frac{-4Q_y}{Q_\alpha} \sin[Q_\alpha(\bar{\tau} - \bar{\tau}_0)] \theta(\bar{\tau} - \bar{\tau}_0) e^{-\nu'_{tot}(\bar{\tau}^3 - \bar{\tau}_0^3) Q_y^2 / 6} \cdot \sum_{\mathbf{Q}'} Q'_\alpha (b_{\mathbf{Q}'}^y b_{\mathbf{Q}-\mathbf{Q}'}^y + v_{\mathbf{Q}'}^y v_{\mathbf{Q}-\mathbf{Q}'}^y) d\bar{\tau}_0$$

$$\begin{aligned} X(0) &= \int_0^\infty \frac{dQ_x}{Q_\alpha} \cos \left[Q_x \left(\sin(\theta) + \cos(\theta) \frac{Q_z}{Q_y} \right) \right] \exp \left[-\frac{\nu'_{tot}}{6} \frac{Q_x^3}{Q_y} \right] \cdot \sum_{\mathbf{Q}'} Q'_\alpha (b_{\mathbf{Q}'}^y b_{\mathbf{Q}-\mathbf{Q}'}^y + v_{\mathbf{Q}'}^y v_{\mathbf{Q}-\mathbf{Q}'}^y) \\ Y(0) &= - \int_0^\infty \frac{dQ_x}{Q_\alpha} \sin \left[Q_x \left(\sin(\theta) + \cos(\theta) \frac{Q_z}{Q_y} \right) \right] \exp \left[-\frac{\nu'_{tot}}{6} \frac{Q_x^3}{Q_y} \right] \cdot \sum_{\mathbf{Q}'} Q'_\alpha (b_{\mathbf{Q}'}^y b_{\mathbf{Q}-\mathbf{Q}'}^y + v_{\mathbf{Q}'}^y v_{\mathbf{Q}-\mathbf{Q}'}^y) \end{aligned}$$

$$I = \int_0^\infty \frac{dQ'_z}{2\pi} \int_{Q_y/2}^\infty \frac{dK_y}{2\pi} \int_{Q_x/2}^\infty \frac{dK_x}{2\pi} Q'_\alpha D(Q'_\alpha) D(Q_\alpha - Q'_\alpha) \quad (3.133)$$

$$\exp \left\{ -\frac{\nu'_{tot}}{6} \left[\frac{(K_x - Q_x/2)^3}{K_y + Q_y/2} + \frac{(K_x + Q_x/2)^3}{K_y - Q_y/2} \right] \right\} \cos(\hat{K}_x - \hat{Q}_x + \Delta\phi)$$

$$\Delta\phi = \phi \left[\sin(\theta) \left(K_y + \frac{Q_y}{2} \right) + \cos(\theta)(Q_z - Q'_z) \right] - \phi \left[\sin(\theta) \left(\frac{Q_y}{2} - K_y \right) + \cos(\theta)Q'_z \right]$$

$$\hat{K}_x = K_x \left(2 \sin(\theta) + \cos(\theta) \frac{Q_z - Q'_z}{K_y + Q_y/2} - \cos(\theta) \frac{Q'_z}{K_y - Q_y/2} \right)$$

$$\hat{Q}_x = Q_x \frac{1}{2} \cos(\theta) \left(\frac{Q_z - Q'_z}{K_y + Q_y/2} + \frac{Q'_z}{K_y - Q_y/2} \right)$$

$$b_y(\bar{\tau}, Q_y, Q_z) = \int_{-\infty}^{\bar{\tau}} \frac{-2Q_y}{Q_\alpha} \sin[Q_\alpha(\bar{\tau} - \bar{\tau}_0)] \theta(\bar{\tau} - \bar{\tau}_0) e^{-\nu'_{tot}(\bar{\tau}^3 - \bar{\tau}_0^3) Q_y^2/6} \cdot I$$

$$I = \int_0^\infty \frac{dQ'_z}{2\pi} \int_{Q_y/2}^\infty \frac{dK_y}{2\pi} \int_{Q_x/2}^\infty \frac{dK_x}{2\pi} \cos(\hat{K}_x) \exp \left\{ -\frac{\nu'_{tot}}{6} \left[\frac{(K_x - Q_x/2)^3}{K_y + Q_y/2} + \frac{(K_x + Q_x/2)^3}{K_y - Q_y/2} \right] \right\}$$

$$Q'_\alpha D(Q'_\alpha) D(Q_\alpha - Q'_\alpha) \left[\cos(\hat{Q}_x) \cos(\Delta\phi) + \sin(\hat{Q}_x) \sin(\Delta\phi) \right] +$$

$$\int_0^\infty \frac{dQ'_z}{2\pi} \int_{Q_y/2}^\infty \frac{dK_y}{2\pi} \int_{Q_x/2}^\infty \frac{dK_x}{2\pi} \sin(\hat{K}_x) \exp \left\{ -\frac{\nu'_{tot}}{6} \left[\frac{(K_x - Q_x/2)^3}{K_y + Q_y/2} + \frac{(K_x + Q_x/2)^3}{K_y - Q_y/2} \right] \right\}$$

$$Q'_\alpha D(Q'_\alpha) D(Q_\alpha - Q'_\alpha) \left[\sin(\hat{Q}_x) \cos(\Delta\phi) + \cos(\hat{Q}_x) \sin(\Delta\phi) \right]$$

$$I = -\frac{1}{2\pi} \int_0^\infty \frac{dQ'_z}{2\pi} \int_{Q_y/2}^\infty \frac{dK_y}{2\pi} Q'_\alpha D(Q'_\alpha) D(Q_\alpha - Q'_\alpha) \left[\cos(\hat{Q}_x) \cos(\Delta\phi) + \sin(\hat{Q}_x) \sin(\Delta\phi) \right]$$

$$\frac{1}{2 \sin(\theta) + \cos(\theta) \left(\frac{Q_z - Q'_z}{K_y + Q_y/2} - \frac{Q'_z}{K_y - Q_y/2} \right)}$$

$$\left\{ \sin \left[\frac{Q_x}{2} \left(2 \sin(\theta) + \cos(\theta) \left(\frac{Q_z - Q'_z}{K_y + Q_y/2} - \frac{Q'_z}{K_y - Q_y/2} \right) \right) \right] + \exp \left[-\frac{q_x^3}{3} \cdot \frac{Q_y}{4K_y^2 - Q_y^2} \right] \right\}$$

$$+ \frac{1}{2\pi} \int_0^\infty \frac{dQ'_z}{2\pi} \int_{Q_y/2}^\infty \frac{dK_y}{2\pi} Q'_\alpha D(Q'_\alpha) D(Q_\alpha - Q'_\alpha) \left[\sin(\hat{Q}_x) \cos(\Delta\phi) + \cos(\hat{Q}_x) \sin(\Delta\phi) \right]$$

$$\frac{1}{2 \sin(\theta) + \cos(\theta) \left(\frac{Q_z - Q'_z}{K_y + Q_y/2} - \frac{Q'_z}{K_y - Q_y/2} \right)}$$

$$\left\{ \cos \left[\frac{Q_x}{2} \left(2 \sin(\theta) + \cos(\theta) \left(\frac{Q_z - Q'_z}{K_y + Q_y/2} - \frac{Q'_z}{K_y - Q_y/2} \right) \right) \right] - 1 \right\}$$

$$\begin{aligned}
I &= \frac{1}{2\pi} \int_0^\infty \frac{dQ'_z}{2\pi} \int_{Q_y/2}^\infty \frac{dK_y}{2\pi} Q'_\alpha D(Q'_\alpha) D(Q_\alpha - Q'_\alpha) \frac{1}{2 \sin(\theta) + \cos(\theta) \left(\frac{Q_z - Q'_z}{K_y + Q_y/2} - \frac{Q'_z}{K_y - Q_y/2} \right)} \\
&\quad \left\{ \sin \left[Q_x \left(\sin(\theta) - \cos(\theta) \frac{Q'_z}{K_y - Q_y/2} \right) \right] - \sin(\hat{Q}_x) + \cos(\hat{Q}_x) \exp \left[-\frac{q_x^3}{3} \cdot \frac{Q_y}{4K_y^2 - Q_y^2} \right] \right\} \cos(\Delta\phi) \\
&\quad \left\{ \cos \left[Q_x \left(\sin(\theta) + \cos(\theta) \frac{Q_z - Q'_z}{K_y + Q_y/2} \right) \right] - \cos(\hat{Q}_x) + \sin(\hat{Q}_x) \exp \left[-\frac{q_x^3}{3} \cdot \frac{Q_y}{4K_y^2 - Q_y^2} \right] \right\} \sin(\Delta\phi) \\
X &= -\frac{1}{2\pi} \int_{-\infty}^0 \frac{dQ_x}{Q_\alpha} \sin \left[Q_x \left(\sin(\theta) + \cos(\theta) \frac{Q_z}{Q_y} \right) \right] \sin \left[Q_x \left(\sin(\theta) - \cos(\theta) \frac{Q'_z}{K_y - Q_y/2} \right) \right] \\
&\quad \exp \left[-\frac{\nu'_{tot}}{6} \cdot \frac{Q_x^3}{Q_y} \right] \cdot \int_0^\infty \frac{dQ'_z}{2\pi} \int_{Q_y/2}^\infty \frac{dK_y}{2\pi} \cos(\Delta\phi) \frac{Q'_\alpha D(Q'_\alpha) D(Q_\alpha - Q'_\alpha)}{2 \sin(\theta) + \cos(\theta) \left(\frac{Q_z - Q'_z}{K_y + Q_y/2} - \frac{Q'_z}{K_y - Q_y/2} \right)} \\
Y &= \frac{1}{2\pi} \int_{-\infty}^0 \frac{dQ_x}{Q_\alpha} \cos \left[Q_x \left(\sin(\theta) + \cos(\theta) \frac{Q_z}{Q_y} \right) \right] \cos \left[Q_x \left(\sin(\theta) + \cos(\theta) \frac{Q_z - Q'_z}{K_y + Q_y/2} \right) \right] \\
&\quad \exp \left[-\frac{\nu'_{tot}}{6} \cdot \frac{Q_x^3}{Q_y} \right] \cdot \int_0^\infty \frac{dQ'_z}{2\pi} \int_{Q_y/2}^\infty \frac{dK_y}{2\pi} \sin(\Delta\phi) \frac{Q'_\alpha D(Q'_\alpha) D(Q_\alpha - Q'_\alpha)}{2 \sin(\theta) + \cos(\theta) \left(\frac{Q_z - Q'_z}{K_y + Q_y/2} - \frac{Q'_z}{K_y - Q_y/2} \right)} \\
X &= -\frac{1}{Q_\alpha} \frac{\Gamma(\frac{4}{3})}{(2\pi)^4} \left(\frac{6Q_y}{\nu'_{tot}} \right)^{1/3} \int_0^\infty dQ'_z \int_{Q_y/2}^\infty dK_y \frac{Q'_\alpha D(Q'_\alpha) D(Q_\alpha - Q'_\alpha) \cos(\Delta\phi) J_c}{2 \sin(\theta) + \cos(\theta) \left(\frac{Q_z - Q'_z}{K_y + Q_y/2} - \frac{Q'_z}{K_y - Q_y/2} \right)} \\
Y &= \frac{1}{Q_\alpha} \frac{\Gamma(\frac{4}{3})}{(2\pi)^4} \left(\frac{6Q_y}{\nu'_{tot}} \right)^{1/3} \int_0^\infty dQ'_z \int_{Q_y/2}^\infty dK_y \frac{Q'_\alpha D(Q'_\alpha) D(Q_\alpha - Q'_\alpha) \sin(\Delta\phi) J_s}{2 \sin(\theta) + \cos(\theta) \left(\frac{Q_z - Q'_z}{K_y + Q_y/2} - \frac{Q'_z}{K_y - Q_y/2} \right)} \\
R &\equiv \frac{\Gamma(\frac{4}{3})}{(2\pi)^4} \left(\frac{6}{\nu'_{tot}} \right)^{1/3}, \quad \tilde{X} \equiv XR, \quad \tilde{Y} \equiv YR, \quad \tilde{D} \equiv DR. \\
\tilde{X} &= -\frac{\sqrt[3]{Q_y}}{Q_\alpha} \int_0^\infty dQ'_z \int_{Q_y/2}^\infty dK_y \frac{[\sin(\theta)(K_y + Q_y/2) + \cos(\theta)Q'_z] J_c}{2 \sin(\theta) + \cos(\theta) \left(\frac{Q_z - Q'_z}{K_y + Q_y/2} - \frac{Q'_z}{K_y - Q_y/2} \right)} D(Q'_\alpha) D(Q_\alpha - Q'_\alpha) \cos(\Delta\phi) \\
\tilde{Y} &= \frac{\sqrt[3]{Q_y}}{Q_\alpha} \int_0^\infty dQ'_z \int_{Q_y/2}^\infty dK_y \frac{[\sin(\theta)(K_y + Q_y/2) + \cos(\theta)Q'_z] J_s}{2 \sin(\theta) + \cos(\theta) \left(\frac{Q_z - Q'_z}{K_y + Q_y/2} - \frac{Q'_z}{K_y - Q_y/2} \right)} D(Q'_\alpha) D(Q_\alpha - Q'_\alpha) \sin(\Delta\phi)
\end{aligned}$$

$$\nu'_{\text{wave}} \approx \frac{R}{(2\pi)^3} I_{3D}, \quad I_{3D} = \int_0^\infty Q_y \tilde{D}_f^2(Q_y, Q_z) dQ_y dQ_z \quad (3.134)$$

$$\begin{aligned} \tilde{D}_i &= \sqrt{\tilde{X}_i^2 + \tilde{Y}_i^2}, \quad \phi_i = \arctan \frac{\tilde{Y}_i}{\tilde{X}_i} \\ D_f &= D_i G(Q_y, Q_z), \quad \phi_f = F(\phi_i) \equiv \arctan \frac{s_{ig}s_u + s_{iu}s_g}{s_{ig}c_u + s_{iu}c_g} \end{aligned} \quad (3.135)$$



SOURCE CODE

In the computer algebra system Maple the confluent Heun function reference is $\text{HeunC}(\alpha, \beta, \gamma, \delta, \eta, z)$. This special function obeys the equation

$$z(z-1)y'' + [Az^2 + Bz + C]y' + [Dz + E]y = 0, \quad (\text{A.1})$$

where

$$A = \alpha, \quad B = 2 + \beta + \gamma - \alpha, \quad C = -1 - \beta, \quad (\text{A.2})$$

$$D = \frac{1}{2}[(2 + \gamma + \beta)\alpha + 2\delta], \quad E = \frac{1}{2}[-\alpha(1 + \beta) + (1 + \gamma)\beta + \gamma + 2\eta]. \quad (\text{A.3})$$

For the series expansion

$$y = \sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} a_n z^n, \quad c_n \equiv a_n z^n \quad (\text{A.4})$$

we arrive at the recursion

$$Fa_{n-1} + Ga_n + Ha_{n+1} = 0, \quad (\text{A.5})$$

where

$$F = \delta + \frac{1}{2}(2n + \beta\gamma), \quad (\text{A.6})$$

$$G = n^2 + (1 + \gamma + \beta - \alpha)n + \frac{1}{2}[\gamma + 2\eta - \alpha + \beta(1 - \alpha + \gamma)]$$

$$H = -(n+1)(n+1+\beta). \quad (\text{A.7})$$

Supposing $a_{-1} = 0$ and $a_0 = 1$ we use the recursion

$$a_{n+1} = -(Fa_{n-1} + Ga_n)/H \quad (\text{A.8})$$

which, for example, gives

$$a_1 = \frac{1}{2} [\beta(1 + \gamma - \alpha) + \gamma + 2\eta - \alpha] / (1 + \beta). \quad (\text{A.9})$$

As the effective Schrödinger equation has a solution for arbitrary $\xi \in (-\infty, +\infty)$, the formal series for Heun function have convergent Padé approximants.

Those Padé approximants can be calculated by the well-known ε -algorithm. First we calculate the series of the partial sums in zeroth (0) approximation

$$S_i^{(0)} = S_{i-1}^{(0)} + c_i \quad (\text{A.10})$$

and for the first N terms we get

$$S_0^{(0)} = c_0, \quad S_1^{(0)} = c_0 + c_1, \quad \dots, \quad S_N^{(0)} = c_0 + c_1 + \dots + c_N. \quad (\text{A.11})$$

Our problem is to calculate the limit of the sequence

$$S = \lim_{n \rightarrow \infty} S_n. \quad (\text{A.12})$$

For this calculation we generate the auxiliary sequence

$$H_i^{(0)} = 1/c_{i+1} \quad (\text{A.13})$$

recalling that $1/0 = 0$, i.e., using pseudoinverse numbers if we have to divide by zero:

$$H_0^{(0)} = 1/c_1, \quad H_1^{(0)} = 1/c_2, \quad \dots, \quad H_{N-1}^{(0)} = 1/c_N. \quad (\text{A.14})$$

The epsilon-algorithm is the calculation of the recursion for the series

$$S_i^{(k)} = S_{i+1}^{(k-1)} + 1 / \left(H_{i+1}^{(k-1)} - H_i^{(k-1)} \right), \quad (\text{A.15})$$

$$H_i^{(k)} = H_{i+1}^{(k-1)} + 1 / \left(S_{i+1}^{(k-1)} - S_i^{(k-1)} \right) \quad (\text{A.16})$$

for all indices for which these relations make sense.

The maximal in modulus auxiliary element $H_{\max} = |H_{I+1}^{(K)}|$ gives the best Padé approximant $S_I^{(K)}$ for the searched limes and the accuracy is of the order of $1/H_{\max}$.

For each accuracy of the final result ϵ we can calculate S and H sequences with an accuracy δ to assure that $1/H_{\max} < \epsilon$. In such a way we obtain a method for calculating the confluent Heun function and the solution to the MHD equation in power series of time. If from physical arguments we know that a solution exists, the divergent series can be summed. Having a method for calculating the confluent Heun function, we can calculate both the even and odd solutions. The accuracy in calculating Heun functions is controlled by the Wronskian

$$W(\psi_g, \psi_u) = \begin{vmatrix} \psi_g(\xi) & \psi_u(\xi) \\ d_\xi \psi_g(\xi) & d_\xi \psi_u(\xi) \end{vmatrix} = 1. \quad (\text{A.17})$$

The constants from the general solution are also given by the Wronskians

$$C_g = W(\psi, \psi_u) = \psi(\xi_0) d_\xi \psi_u(\xi_0) - \psi_u(\xi_0) d_\xi \psi(\xi_0), \quad (\text{A.18})$$

$$C_u = W(\psi_g, \psi) = \psi_g(\xi_0) d_\xi \psi(\xi_0) - \psi(\xi_0) d_\xi \psi_g(\xi_0). \quad (\text{A.19})$$

Those formulae generally apply for a Cauchy problem where the initial conditions are imposed on the function being sought, i.e., on $\psi(\xi_0)$ and its derivative $d_\xi \psi_g(\xi_0)$.

```

////////////////////////////////////
//
// After the Fortran90 program by E. Penev published in
// T. Mishonov and E. Penev
// "Thermodynamics of Gaussian fluctuations and paraconductivity in
// layered superconductors" International Journal of
// Modern Physics B, Vol. 14, No. 32 (2000) 3831-3879.
//
// Former C version of this program is given in the preprint:
// T.M. Mishonov, S.I. Klenov, E.S. Penev,
// "Temperature dependence of the specific heat and the penetration
// depth of anisotropic-gap BCS superconductors for a factorizable
// pairing potential" http://arxiv.org/abs/cond-mat/0212491v5
// [cond-mat.supr-con] [v5] Wed, 28 Apr 2004
//
// Owners: Todor M. Mishonov & Zlatan D. Dimitrov
//
// Description:
// Finds the limit of a series in the case where only
// the first N+1 terms b[i] are known.
//
// Method:
// The routine operates by applying the epsilon-algorithm
// to the sequence of partial sums of a series supplied on input.
// For description of the algorithm, please see
// G. Baker, Jr., and P. Graves-Morris, Pad Approximants,
// G.-C. Rota editor,
// Encyclopedia of Mathematics and its Applications,
// Vol. 13, (Addison-Wesley, London, 1981), Table 3, p. 78;
// C. Brezinski, Pade-type Approximation and General Orthogonal
// Polynomials (Birkhauser, 1980);
// P. Wynn, Math. Tables Aids Comput. 10, 91 (1956);
// D. Shanks, J. Math. Phys. 34, 1 (1955).
//
////////////////////////////////////

```



```

for(i=0;i<=n;i++) norma+=fabs(a[i]);
if(norma==0) {ierr=2; strcpy(report,"norma==0"); goto label;}
// this line is "insgurance" against all terms to be == 0

if( fabs(a[0]) > fabs(a[n]) ) for(i=0;i<=n;i++)
    {for(temp=0,k=i;k>=0;temp+=a[k],s[i]=temp,k--);}
    else for(s[0]=a[0],i=1;i<=n;i++) s[i]=s[i-1]+a[i];
    // summation starts from smallest terms
// implicitey we suppose s[-1]=0; the s[0]=s[-1]+a[0];
// then a[i]=s[i]-s[i-1] for i=0,1,2,3, ...
// in Fortran program Limes the array s[i] is the input variable

for(i=1;i<=n;i++)
{
if( a[i] != 0 ) { nz=0; if( 1/fabs(a[i])>bmax ) {
bmax=1/fabs(a[i]); err=fabs(a[i]); iPade=i;aLimes=s[i];ierr=1;
strcpy(report,"Usual Taylor summation");
}
}
if( a[i] == 0 ) {nz++; if( nz == mnz )
{aLimes=s[i-mnz]; iPade=i-nz, err=0; ierr=3;
strcpy(report,"a[i]==0,..., a[i-mnz] ==0; polynom?; no need of epsilon
goto label;}}
}
// the series could be fastly covergent -- no need of epsilon
// algorith the label means exit the end of the routine

for(i=0;i<=mnz;i++) scale+=fabs(a[i]);
for(i=0;i<=n;i++){ if(fabs(a[i])>scale/eps) {n=i; break;}}
// good working line

//printf("n=%i, ierr=%i \n",n,ierr);
if(n<3) {ierr=5; strcpy(report,"n<3"); aLimes=s[n]; iPade=n; goto label;}
// this line is an "insgurance" for small n
// all partial summs sould be "comparable" within mashibe epsilon;
// a new technolosical detail

// a lot of unnecessary details above!

// INITIALIZATION
for(i=0;i<=n;t[0][i]=s[i],i++);// epsilon table

```

```

for(i=0;i<=n-1;i++)
{
if(a[i+1] != 0) {b[i]=1/a[i+1];
                t[1][i]=b[i]; // epsilon table
                }
else {b[i]=0;
      t[1][i]=b[i]; // epsilon table
      ndz++;
      ierr= 4;
      strcpy(report,"if( a[i+1] == 0) then 1/0=0");}
      // should be harmless division by zero
}

// BEGINING OF EPSILON ALGORITHM
// epsilon algorithm beginning with 2nd row: (E-W)*(S-N) ==1;
imax=n-2;
for(k=2;k<=n+1;k++) // odd k means pade row s[i]; even k help row b[i]
{

if(k%2==0) { for(i=0;i<=imax;i++)
{s[i]=s[i+1];
                ew=b[i+1]-b[i];
                if(ew !=0) s[i]+=1/ew;
                t[k][i]=s[i]; // epsilon table
            }
        }

if(k%2==1) { for(i=0;i<=imax;i++)
{b[i]=b[i+1];
        ew=s[i+1]-s[i];
        if(ew !=0) b[i]+=1/ew;
                t[k][i]=b[i]; // epsilon table
                if ( bmax < fabs(b[i]) ) {
bmax=fabs(b[i]);
        // maximal help row element criterion; the MAIN detail!
aLimes=s[i];
kPade=(k-1)/2;
iPade=i+kPade;
err=1/bmax;
ierr=0;
strcpy(report,"ierr=0;The regular work of epsilon algorithm");
        }
}
}

```

```

    }

}

imax--;
if(imax<0) break;
}
label: nPade=iPade+kPade+1;// label for emrgency exit
if(err/norma<epsf) {fLimes= (float) aLimes; iOK=1;}
printf("norma>>>%Le\n",norma);
puts(report);
printf("iPade=%i,kPade=%i,err=%Le,ndz=%i,nz=%i,i=%i,nPade=%i,n=%i\n",
iPade,kPade,err,ndz,nz,i,nPade,n);
printf(">>>>iOK=%i<<<<\n",iOK);
*iP=iPade;
*kP=kPade;
*p_err=err;
*pLimes=aLimes;
*p_OK=iOK;
*p_nk=nk;
*p_ierr=ierr;
return aLimes;
}

////////////////////////////////////
// FUNCTION

long double function (long double x)
{
int n=51,iPade, *p=&iPade ; //n=7 min pri sdvoiavane(mdl=2) ili n=3
int kPade,iOK,nk,ierr;
long double result, err,aLimes,sign;
long double a[99];
int i=0;

/////
long double v[100],suma=0;
/long double alpha=0,beta=-0.5,gamma=0,delta=-0.025,eta=0.275,az=0,buki=0
/////

//HeunC(alpha,beta,gamma,delta,eta;Z)

```

```

/*v[0]=1;
v[1]=(gamma - alpha + beta*(gamma-alpha+1) + 2*eta)/(2*(beta+1));
for(i=2;i<n+1;i++)
{
m=i-1;
az=delta + (alpha*0.5)*(m+m + beta + gamma);
buki=m*(m+1+beta+gamma-alpha) + 0.5*(gamma+eta+eta-alpha+beta*(1-alpha+ga
vedi=-(m+1)*(m+1+beta);
v[i]=-(v[i-1]*buki + v[i-2]*az)/vedi;
}

*/
sign=1;
v[0]=0;
for(i=1;i<=n;i++)
{
v[i]=sign/i;
sign=-sign;
//printf("v[%i]=%Lg\n",i,v[i]);
}

long double xn;
xn=1;
for(i=0;i<=n;i++)
{
a[i]=v[i]*xn;xn*=x;
//printf("a[%i]=%Lg\n",i,a[i]);
suma+=a[i];
}
//printf("\n suma= %.12Lf\n ",suma);
// ln(1+x)=x-x^2/2+x^3/3-x^4/4+x^5/5-...

/*
// here cut %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
int mdl=2;// number of unified terms
int nk=n/mdl;
for(k=0;k<=nk;k++){
    b[k]=0;
    for(i=0;i<mdl;i++)
        {b[k]+=a[mdl*k+i];}
}

```



```
        }
// here cut %%%%%%%%%%%%%%%
*/

result=summa(n,a,p,&kPade,&err,&aLimes,&iOK,&nk,&ierr);

return result;

}


int main(void)
{
long double x,z;

x=exp(1)-1;
//for(x=0.1;x<15;x+=0.1)
//{
z=function(x); //z=ln(e)=1;
printf("%Lg %.12Lf\n",x,z);
printf("ln(1+x)=%.12f\n",log(M_E));
//}

return 0;
}
```


INTEGRAL EQUATIONS

```

#include <stdio.h>
#include <math.h>

double Q0=M_PI/2;

double deltag( double q)
{
return ( -M_PI/2. + atan(fabs(q)/Q0) );
}

double deltau( double q)
{
return ( -M_PI/2. );
}

int main(void)
{
int N=500,N1=N+1,i,j,k,ja,ka,iter,Niter=65;
double Qmax=5.,DQ=Qmax/N, Q,P,X,Y,XQ,YQ,x,y,sig,siu,aN,rN=1.,rNnew,rN2=1.
double D[N1],D_i[N1],phi[N1],phi_i[N1],c[N1],s[N1],q[N1],dg[N1],du[N1],sg

    for(i=1;i<=N;i++)
    {
q[i]=DQ*i;

```

```

dg[i]=deltag(q[i]);
    sg[i]=sin(dg[i]);
    cg[i]=cos(dg[i]);
    du[i]=deltau(q[i]);
    su[i]=sin(du[i]);
    cu[i]=cos(du[i]);
    sug[i]=sin(du[i]-dg[i]);
}

for(i=1;i<=N;i++)
{
D[i]=1.;
phi[i]=0.;
    c[i]=cos(phi[i]);
    s[i]=sin(phi[i]);
}

for(iter=0;iter<=Niter;iter++)// beginning of iteration procedure
{
for(i=1;i<=N;i++)
    { // tova e cycle po Q
        Q=q[i];
        XQ=0.; YQ=0.;
        for(j=-N;j<=N;j++)
            { // tova e cycle po P in (-infty,+infty)
                ja=abs(j);
                P=DQ*j;
                k=i-j; ka=abs(k); //QP=Q-P=DQ*(i-j)=DQ*k
                if(ka>N) continue;
                if(ka==0) continue; // i,ja,ka in [1,N]
                if(ja==0) continue; // i,j,k in [-N, -1] U [1, N]
                XQ+=P*D[ja]*D[ka]*(c[ja]*c[ka]+s[ja]*s[ka]); // *DQ
                YQ+=P*D[ja]*D[ka]*(s[ja]*c[ka]-c[ja]*s[ka]); // *DQ
            } // next j
        d_reg=pow(fabs(Q),2./3);
        X=(XQ*DQ)/ d_reg; // *DQ
        Y=(YQ*DQ)/ d_reg; // *DQ
        D_i[i]=sqrt(X*X+Y*Y); // Dnew_initial
        phi_i[i]=atan2(Y,X);
        sig=sin(phi_i[i]-dg[i]);
        siu=sin(phi_i[i]-du[i]);
    }
}

```

```

x=sig*cos(du[i])+sin(phi_i[i]-du[i])*cos(dg[i]);
y=sin(phi_i[i]-dg[i])*sin(du[i])+siu*sin(dg[i]);
phi[i]=atan2(y,x);
aN=(sig*su[i]+siu*sg[i])*(sig*su[i]+siu*sg[i])
    +(sig*cu[i]+siu*cg[i])*(sig*cu[i]+siu*cg[i]); // a Numerator
G[i]=sqrt(aN)/fabs(sug[i]);
Dnew[i]=G[i]*D_i[i];
} // next i

rN2=0.; // real norma D
for(i=1;i<=N;i++) {rN2+=Dnew[i]*Dnew[i]*DQ;}
rNnew=sqrt(rN2);
if(iter==Niter-1){err2=(rNnew-1/zN)*2/(rNnew+1/zN);}
err1=(rNnew-rN)*2./(rNnew+rN);
rN=rNnew;

for(i=1;i<=N;i++) {Dnew[i]/=rN;} // Normalization. Now |Dnew|=1

for(i=1;i<=N;i++)
{
    if(iter == Niter-1) { err3+=fabs(D[i]-Dnew[i])/fabs(D[i]); } //err3+=f
    D[i]=Dnew[i];
    c[i]=cos( phi[i]);
    s[i]=sin( phi[i]);
    } // getting old

    if(iter==Niter-1) {zN=rN; for(i=1;i<=N;i++) D[i]/=zN; D_i[i]=D[i]

} // next iter

//for(i=1;i<=N;i++)
//printf("%g %g\n",q[i],D[i]*D[i]*q[i]);
//printf("%g %g\n",q[i],D[i]*D[i]*q[i] );
//printf("%g %g\n",q[i],exp(0.69286746320344-40.915167099567*q[i]));

for(rID=0.,Qav=0., i=1;i<=N;i++)
{
Sx[i]=D[i]*D[i]*q[i];
rID+=Sx[i]*DQ;

```

```
Qav+=q[i]*Sx[i]*DQ;  
//printf("%g %g\n",q[i],D[i]*q[i]);  
}  
Qav/=rID;  
printf("Niter=%i  rID=%g  Qav=1/%g,  Qav/DQ=%g,  err3=%g \n",Niter,  
  
return 0;  
}
```



FAST OSCILLATION FUNCTION

$$\begin{aligned}
 I &= \int_0^\infty \cos(kx) f(x) dx = \frac{1}{k} \int_0^\infty f(x) d \sin(kx) = \frac{1}{k} f(x) \sin(kx) \Big|_0^\infty - \frac{1}{k} \int_0^\infty \sin(kx) df(x) \\
 &= -\frac{1}{k} \int_0^\infty f'(x) \sin(kx) dx = \frac{1}{k^2} \int_0^\infty f'(x) d \cos(kx) = \frac{1}{k^2} f'(x) \cos(kx) \Big|_0^\infty - \frac{1}{k^2} \int_0^\infty \cos(kx) f''(x) dx \\
 &= -\frac{f'(0)}{k^2} - \frac{1}{k^2} \int_0^\infty \cos(kx) f''(x) dx = \int_0^\infty \cos(kx) f(x) dx \quad (\text{C.1})
 \end{aligned}$$

$$\begin{aligned}
 I &= -\frac{f'(0)}{k^2} - \frac{1}{k^2} \left[-\frac{f'''(0)}{k^2} - \frac{1}{k^2} \int_0^\infty \cos(kx) f^{(4)}(x) dx \right] \\
 &= \frac{f'(0)}{k^2} + \frac{1}{k^2} \left[-\frac{f^{(3)}(0)}{k^2} + \frac{1}{k^2} \left\{ \frac{f_0^{(5)}}{k^2} + \frac{1}{k^2} \int_0^\infty \cos(kx) f^{(6)}(x) dx \right\} \right] \\
 &= \frac{f_0^{(1)}}{k^2} + \frac{f_0^{(3)}}{(k^2)^2} - \frac{f_0^{(5)}}{(k^2)^3} + \frac{f_0^{(7)}}{(k^2)^4} - \frac{f_0^{(9)}}{(k^2)^5} + \dots \\
 &= -\frac{1}{k^2} \left[1 - \frac{1}{(k^2)^1} d_x^2 + \frac{1}{(k^2)^2} (d_x^2)^2 - \frac{1}{(k^2)^3} (d_x^2)^3 + \dots \right] d_x f(x) \Big|_0 \quad (\text{C.2})
 \end{aligned}$$

$$\frac{1}{1+q} = 1 - q + q^2 - q^3 + q^4 - q^5 \quad (\text{C.3})$$

$$I = -\frac{1}{k^2} \left[\frac{1}{1 + \frac{d_x^2}{k^2}} \right] d_x f(x) \Big|_0 = -\left[\frac{d_x}{k^2 + d_x^2} \right] f(x) \Big|_0 \quad (\text{C.4})$$

Example:

$$f(x) = e^{-\alpha x}, \quad d_x f(x) = -\alpha f(x), \quad d_x^n f(x) = (-\alpha)^n f(x) \quad (\text{C.5})$$

$$\frac{1}{k^2 + d_x^2} e^{-\alpha x} = \frac{1}{k^2 + (-\alpha)^2} e^{-\alpha x} \quad (\text{C.6})$$

$$\begin{aligned} J &= \int_0^\infty \sin(kx) f(x) dx = -\frac{1}{k} \int_0^\infty f(x) d \cos(kx) = -\frac{1}{k} f(x) \cos(kx) \Big|_0^\infty + \frac{1}{k} \int_0^\infty \cos(kx) d f(x) \\ &= \frac{f_0}{k} + \frac{1}{k^2} \int_0^\infty f'(x) d \sin(kx) = -\frac{1}{k^2} f'(x) \sin(kx) \Big|_0^\infty + \frac{1}{k^2} \int_0^\infty \sin(kx) f''(x) dx \\ &= \frac{f'(0)}{k^2} + \frac{1}{k^2} \int_0^\infty \sin(kx) f''(x) dx = -\int_0^\infty \sin(kx) f(x) dx \quad (\text{C.7}) \end{aligned}$$

$$\begin{aligned} J &= -\frac{f'(0)}{k^2} - \frac{1}{k^2} \left[-\frac{f'''(0)}{k^2} - \frac{1}{k^2} \int_0^\infty \cos(kx) f^{(4)}(x) dx \right] \\ &= \frac{f'(0)}{k^2} + \frac{1}{k^2} \left[-\frac{f^{(3)}(0)}{k^2} + \frac{1}{k^2} \left\{ \frac{f_0^{(5)}}{k^2} + \frac{1}{k^2} \int_0^\infty \cos(kx) f^{(6)}(x) dx \right\} \right] \\ &= \frac{f_0^{(1)}}{k^2} + \frac{f_0^{(3)}}{(k^2)^2} - \frac{f_0^{(5)}}{(k^2)^3} + \frac{f_0^{(7)}}{(k^2)^4} - \frac{f_0^{(9)}}{(k^2)^5} + \dots \\ &= -\frac{1}{k^2} \left[1 - \frac{1}{(k^2)^1} d_x^2 + \frac{1}{(k^2)^2} (d_x^2)^2 - \frac{1}{(k^2)^3} (d_x^2)^3 + \dots \right] d_x f(x) \Big|_0 \quad (\text{C.8}) \end{aligned}$$

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